AN ELEMENTARY TREATISE

ON THE

DYNAMICS OF A SYSTEM OF RIGID BODIES.

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AN ELEMENTARY TREATISE

ON THE

DYNAMICS OF A SYSTEM OF RIGID BODIES.

With numerous Examples.

BY

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PREFACE.

The name of Mr C. A. Smith, B.A. of St Peter's College, has been joined with mine in the advertisements of this Treatise. His serious illness and absence abroad have, however, unfortunately deprived me of his assistance, and have rendered it necessary that I should undertake the work alone.

The numerous Examples, which will be found at the end of each chapter, have been chiefly selected from the Examination Papers set in the University and in the Colleges during the last few years.

I have also to express my acknowledgments to Mr M^{c}Dowell of Pembroke College, for his assistance in correcting the proof sheets, and I believe that there are few errors which have escaped detection.

EDWARD J. ROUTH.

PETERHOUSE,
Oct. 20, 1860.

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DYNAMICS OF A RIGID BODY.

CHAPTER I.

ON MOMENTS OF INERTIA.

SECT. I. *Elementary Properties.*

1. DEF. 1. If the mass of every particle of a material system be multiplied by the square of its distance from a straight line, the sum of the products so formed is called the *moment of inertia* of the system about that line.

DEF. 2. If M be the mass of a system, and κ be such a quantity that $M\kappa^2$ is its moment of inertia about a given straight line, then κ is called the *radius of gyration* of the system about that line.

DEF. 3. If two straight lines Ox, Oy be taken as axes, and if the mass of every particle of the system be multiplied by its *two* co-ordinates x, y, the sum of the products is called the *product of inertia* of the system about those two axes.

Let a body be referred to any rectangular axes Ox, Oy, Oz, meeting in a point O, and let x, y, z be the co-ordinates of any particle m, then, according to these definitions, the moments of inertia about the axes of x, y, z respectively will be

$$\left.\begin{aligned} A &= \Sigma m\,(y^2+z^2) \\ B &= \Sigma m\,(z^2+x^2) \\ C &= \Sigma m\,(x^2+y^2) \end{aligned}\right\},$$

and the products of inertia about the axes of yz, zx, xy,

$$D = \Sigma m(yz),$$
$$E = \Sigma m(zx),$$
$$F = \Sigma m(xy).$$

2. In the particular case of the body being a lamina, taking the axis of z normal to the lamina, we have $z=0$, and therefore

$$A = \Sigma my^2, \qquad B = \Sigma mx^2,$$
$$C = \Sigma m(x^2 + y^2).$$

Hence $C = A + B$, or the moment of inertia of a lamina about an axis perpendicular to its plane is equal to the sum of the moments of inertia about any two perpendicular axes in its plane drawn from the point where the former axis meets the plane.

3. Prop. I. *Given the moments and products of inertia about all axes through the centre of gravity of a body, to deduce the moments and products about all other parallel axes.*

"The moment of inertia of a body or system of bodies about any axis is equal to the moment of inertia about a parallel axis through the centre of gravity plus the moment of inertia of the whole mass collected at the centre of gravity about the original axis."

"The product of inertia about any two axes is equal to the product of inertia about two parallel axes through the centre of gravity plus the product of inertia of the whole mass collected at the centre of gravity about the original axis."

First, take the axis about which the moment of inertia is required as the axis of z. Let m be the mass of any particle of the body, which generally will be any small element.

Let x, y, z be the co-ordinates of m,

$\bar{x}$, $\bar{y}$, $\bar{z}$ those of the centre of gravity G of the whole system of bodies,

x', y', z' those of m referred to a system of parallel axes through the centre of gravity.

Then since

$$\frac{\Sigma mx'}{\Sigma m}, \quad \frac{\Sigma my'}{\Sigma m}, \quad \frac{\Sigma mz'}{\Sigma m}$$

are the co-ordinates of the centre of gravity of the system referred to the centre of gravity as the origin, it follows that

$$\Sigma mx' = 0, \quad \Sigma my' = 0, \quad \Sigma mz' = 0.$$

The moment of inertia of the system about the axis of z is

$$\begin{aligned} &= \Sigma m (x^2 + y^2), \\ &= \Sigma m \{(\bar{x} + x')^2 + (\bar{y} + y')^2\}, \\ &= \Sigma m (\bar{x}^2 + \bar{y}^2) + \Sigma m (x'^2 + y'^2), \\ &\qquad + 2\bar{x} . \Sigma mx' + 2\bar{y} . \Sigma my'. \end{aligned}$$

Now $\Sigma m (\bar{x}^2 + \bar{y}^2)$ is the moment of inertia of a mass Σm collected at the centre of gravity, and $\Sigma m (x'^2 + y'^2)$ is the moment of inertia of the system about an axis through G, also $\Sigma mx' = 0$, $\Sigma my' = 0$; whence the proposition is proved.

Secondly, take the axes of x, y as the axes about which the product of inertia is required.

The product required is

$$\begin{aligned} &= \Sigma m\, xy, \\ &= \Sigma m (\bar{x} + x')(\bar{y} + y'), \\ &= \bar{x}\bar{y} . \Sigma m + \Sigma m (x'y') \\ &\qquad + \bar{x}\Sigma my' + \bar{y}\Sigma mx' \\ &= \bar{x}\bar{y}\Sigma m + \Sigma mx'y'. \end{aligned}$$

Now $\bar{x}\bar{y} . \Sigma m$ is the product of inertia of a mass Σm collected at G and $\Sigma mx'y'$ is the product of the whole system about axes through G; whence the proposition is proved.

4. Let there be two parallel axes A and B at distances a and b from the centre of gravity of the body. Then, if M be the mass of the material system,

$$\left.\begin{matrix}\text{moment of inertia} \\ \text{about } A\end{matrix}\right\} - Ma^2 = \left\{\begin{matrix}\text{moment of inertia} \\ \text{about } B\end{matrix}\right. - Mb^2.$$

Hence when the moment of inertia of a body about one axis is known, that about any other parallel axis may be found. It is obvious that a similar proposition holds with regard to the products of inertia.

5. The preceding proposition may be generalised as follows. Let any system be in motion, and let x, y, z be the co-ordinates at time t of any particle of mass m, then $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ are the velocities, and $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$ the accelerations of the particle resolved parallel to the axes. Suppose

$$V=\Sigma m\phi\left(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \; y, \frac{dy}{dt}, \frac{d^2y}{dt^2}, \; z, \frac{dz}{dt}, \frac{d^2z}{dt^2}\right)$$

to be a given function depending on the structure and motion of the system, the summation extending throughout the system. Also let ϕ be an algebraic function of the first or second order. Thus ϕ may consist of such terms as

$$Ax^2 + Bx\frac{dy}{dt} + C\overline{\frac{dz}{at}}\Big]^2 + Eyz + Fx + \ldots\ldots\ldots$$

where A, B, C, &c. are some constants. Then the following general principle will hold.

"The value of V for any system of co-ordinates is equal to the value of V obtained for a parallel system of co-ordinates with the centre of gravity for origin plus the value of V for the whole mass collected at the centre of gravity with reference to the first system of co-ordinates."

For let $\bar{x}, \bar{y}, \bar{z}$ be the co-ordinates of the centre of gravity, and let

$$x = \bar{x} + x', \quad \text{\&c. \&c.};$$

$$\therefore \frac{dx}{dt} = \frac{d\bar{x}}{dt} + \frac{dx'}{dt}, \text{ \&c. \&c.}$$

Now since ϕ is an algebraic function of the second order of

$x, \frac{dx}{dt}, \frac{d^2x}{dt^2}$; y, &c. it is evident that on making the above substitution and expanding, the process of squaring &c. will lead to three sets of terms, those containing only $\bar{x}, \frac{d\bar{x}}{dt}, \frac{d^2\bar{x}}{dt^2}$, &c., those containing the products of $\bar{x}, x'$, &c., and lastly those containing only $x', \frac{dx'}{dt}$, &c. The first of these will on the whole make up $\phi\left(\bar{x}, \frac{d\bar{x}}{dt}, \text{\&c.}\right)$, and the last $\phi\left(x', \frac{dx'}{dt}, \text{\&c.}\right)$. Hence we have

$$V = \Sigma m\phi\left(\bar{x}, \frac{d\bar{x}}{dt} + \dots\right) + \Sigma m\phi\left(x', \frac{dx'}{dt} + \dots\right)$$

$$+ \Sigma m\left(A\bar{x}\frac{dx'}{dt} + B\frac{d\bar{x}}{dt}x' + C\bar{x}\frac{dy'}{dt} + \dots\right),$$

where A, B, C, &c. are some constants.

Now the term $\Sigma m\left(\bar{x}\frac{dx'}{dt}\right)$ is the same as $\bar{x}\Sigma m\frac{dx'}{dt}$, and this vanishes. For since $\Sigma mx' = 0$, it follows that

$$\Sigma m\frac{dx'}{dt} = 0.$$

Similarly all the other terms in the second line vanish.

Hence the value of V is reduced to two terms. But the first of these is the value of V at the origin for the whole mass collected at the centre of gravity, and the second of these the value of V for the whole system referred to the centre of gravity as origin. Hence the proposition is proved.

The proposition would obviously be true if

$$\frac{d^3x}{dt^3}, \frac{d^3y}{dt^3}, \frac{d^3z}{dt^3},$$

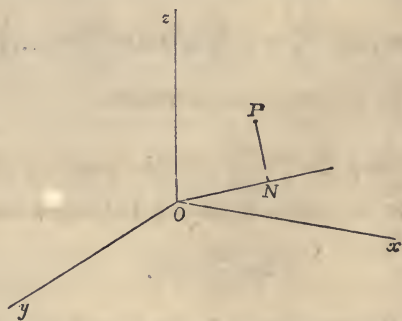
or any higher differential coefficients were also present in the function V.

6. Prop. II. *Given the moments and products of inertia about three straight lines at right angles meeting in a point, to deduce the moments and products of inertia about all other axes meeting in that point.*

"Take these three straight lines as the axes of co-ordinates. Let A, B, C be the moments of inertia about the axes of x, y, z; D, E, F the products of inertia about the axes of yz, zx, xy. Let α, β, γ be the direction cosines of any straight line through the origin, then the moment of inertia Q of the body about that line will be given by the equation

$$Q = A\alpha^2 + B\beta^2 + C\gamma^2 - 2D\beta\gamma - 2E\gamma\alpha - 2F\alpha\beta."$$

Let P be any point of the body at which a mass m is situated, and let x, y, z be the co-ordinates of P. Let ON be



the line whose direction cosines are α, β, γ, draw PN perpendicular to ON.

Since ON is the projection of OP, it is clearly

$$= x\alpha + y\beta + z\gamma,$$

also $$OP^2 = x^2 + y^2 + z^2,$$

and $$1 = \alpha^2 + \beta^2 + \gamma^2.$$

The moment of inertia Q about ON

$$= \Sigma m\, PN^2$$

$$= \Sigma m \{x^2 + y^2 + z^2 - (\alpha x + \beta y + \gamma z)^2\}$$

$$= \Sigma m \{(x^2+y^2+z^2)(\alpha^2+\beta^2+\gamma^2) - (\alpha x+\beta y+\gamma z)^2\}$$

$$= \Sigma m (y^2+z^2)\alpha^2 + \Sigma m (z^2+x^2)\beta^2 + \Sigma m (x^2+y^2)\gamma^2$$
$$- 2\Sigma myz.\beta\gamma - 2\Sigma mzx.\gamma\alpha - 2\Sigma mxy.\alpha\beta$$

$$= A\alpha^2 + B\beta^2 + C\gamma^2 - 2D\beta\gamma - 2E\gamma\alpha - 2F\alpha\beta.$$

7. This result may be exhibited geometrically; for construct the ellipsoid whose equation is

$$AX^2 + BY^2 + CZ^2 - 2DYZ - 2EZX - 2FXY = \epsilon^4.$$

Then if R represent the length of any radius vector from the centre whose direction cosines are α, β, γ,

$$X = R\alpha, \quad Y = R\beta, \quad Z = R\gamma;$$

substituting, we have $Q = \dfrac{\epsilon^4}{R^2}$. Whence the moment of inertia about any radius vector from the centre varies inversely as the square of that radius vector.

If this ellipsoid be referred to any other set of axes through its centre, the coefficients will be the moments and twice the products of inertia about the new axes. For take the Polar Equation

$$\frac{\epsilon^4}{R^2} = A\alpha^2 + B\beta^2 + C\gamma^2 - 2D\beta\gamma - 2E\gamma\alpha - 2F\alpha\beta,$$

and compare it with the general expression for the moment of inertia about the line whose direction cosines are α, β, γ. Then, since the two results must be the same for all values of α, β, γ, the geometrical meaning of A, B, C, &c. is evident.

Also, if the surface be referred to its principal diameters as axes, its equation will be of the form

$$AX^2 + BY^2 + CZ^2 = \epsilon^4,$$

and A, B, C being moments of inertia are essentially positive. Hence the surface is an ellipsoid.

Every point of a material system has therefore its corresponding ellipsoid whose centre is situated at that point.

This is called the Momental Ellipsoid at that point. It is also sometimes called Poinsot's Ellipsoid. When the momental ellipsoid at any point is determined, the moment of inertia about any radius vector from the centre is proportional to the inverse square of that radius vector, and the relations of these several moments of inertia to each other may be deduced from the corresponding relations of the radii vectores of an ellipsoid.

8. The properties of the products of inertia of a body about different sets of axes are not so useful as to require a complete discussion. The reader will have no difficulty in deducing the following results from the properties of an ellipsoid.

(1) If any point O be given and any plane drawn through it, then two straight lines at right angles Ox, Oy, can always be found such that the product of inertia about these lines is zero.

These are the axes of the section of the momental ellipsoid at the point O formed by the given plane.

(2) If two other straight lines at right angles Ox', Oy' be taken in the same plane, making an angle θ measured in the positive direction with Ox, Oy respectively, then the product of inertia F' about Ox', Oy' is given by the equation

$$F' = \tfrac{1}{2}\sin 2\theta\,(A - B),$$

where A and B are the moments of inertia about Ox, Oy.

(3) If A', B' be the moments of inertia about Ox', Oy', then the expression $A'B' - F'^2$ is constant for all positions of Ox', Oy' in the same plane, and is therefore equal to AB.

(4) The value of $A'B' - F'^2$ is the same for all planes through O such that the areas of the sections formed by them with the momental ellipsoid at O is constant.

(5) For any plane whose equation is

$$lx + my + nz = 0,$$

the value of the product of inertia is given by the equation

$$A'B' - F'^2 = B_0C_0l^2 + C_0A_0m^2 + A_0B_0n^2,$$

where A_0, B_0, C_0 are the moments of inertia about the principal diameters of the ellipsoid whose centre is at O, and l, m, n the direction cosines of the plane referred to these diameters as axes.

(6) If I be the moment of inertia about any line in this plane making an angle θ with Ox, then

$$I = A\cos^2\theta + B\sin^2\theta.$$

For the section of the momental ellipsoid by the plane is the ellipse whose equation, referred to its principal diameters as axes, is

$$Ax^2 + By^2 = \epsilon^4,$$

whence the property follows at once.

9. Def. When three straight lines at right angles and meeting in a given point are such that if they be taken as axes of co-ordinates the products

$$\Sigma mxy, \quad \Sigma myz, \quad \Sigma mzx$$

all vanish, these are said to be Principal Axes.

Principal axes can only be defined as a system, and not separately.

The moments of inertia about the principal axes at any point are sometimes called the Principal Moments of Inertia at that point.

10. Prop. III. *At every point of a material system there are always three principal axes at right angles to each other.*

The products of inertia about the axes are half the coefficients of XY, YZ, ZX in the equation to the momental ellipsoid referred to these straight lines as axes of co-ordinates. Now if an ellipsoid be referred to its principal diameters as axes, these coefficients vanish. Hence the principal diameters of the ellipsoid *are* the principal axes of the system. But every ellipsoid has at least three principal diameters, hence every material system has at least three principal axes.

If A, B, C be the three principal moments of inertia at any point, the expression given in Prop. II. for the moment of inertia about any other line becomes

$$Q = A\alpha^2 + B\beta^2 + C\gamma^2.$$

11. The positions of the principal axes at many points in a body may also be found by inspection.

Thus the principal axes of an ellipsoid at the centre of gravity are its principal diameters. Let these be taken as axes, then the sum $\Sigma mxy = 0$. For if any element m be taken two of whose co-ordinates are x, y, another element m, of equal mass, can be found whose corresponding co-ordinates are $-x$, y. Hence the above sum for the whole body is zero. Similarly $\Sigma myz = 0$, $\Sigma mzx = 0$, and these, by Art. 9, are the conditions that the diameters are the principal axes at the centre.

So also the principal axes at the centre of an ellipse are the two principal diameters and a normal to the plane of the ellipse.

By a consideration of some simple properties of ellipsoids, the following propositions are evident:

Of the moments of inertia of a body about axes meeting at a given point, the moment of inertia about one of the principal axes is greatest and about another the least.

For, in the momental ellipsoid, the moment of inertia about any radius vector from the centre is least when that radius vector is greatest and vice versa. And it is evident that the greatest and least radii vectores are two of the principal diameters.

If the three principal moments at any point O be equal to each other, the ellipsoid becomes a sphere. Every diameter is then a principal diameter, and the radii vectores are all equal. Hence every straight line through O is a principal axis at O, and the moments of inertia about them are all equal.

For example, the perpendiculars from the centre of gravity of a cube on the three faces are principal axes; for, the body being referred to them as axes, we clearly have $\Sigma mxy = 0$, $\Sigma myz = 0$, $\Sigma mzx = 0$. Also the three moments of inertia about them are by symmetry equal. Hence every axis through the centre of gravity of a cube is a principal axis, and the moments of inertia about them are all equal.

12. The reciprocal surface of the momental ellipsoid is another ellipsoid, which is called the Ellipsoid of Gyration. This second surface has also been employed to represent, geometrically, the position of the principal axes and the moment of inertia about any line.

"To show that the reciprocal surface of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is the ellipsoid

$$a^2x^2 + b^2y^2 + c^2z^2 = \epsilon^4."$$

Let ON be the perpendicular from the origin O on the tangent plane at any point P of the first ellipsoid, and let l, m, n be the direction cosines of ON, then

$$ON^2 = a^2l^2 + b^2m^2 + c^2n^2.$$

Produce ON to Q so that $OQ = \frac{\epsilon^2}{NO}$, then Q is a point on the reciprocal surface. Let $OQ = R$;

$$\therefore \frac{\epsilon^4}{R^2} = a^2l^2 + b^2m^2 + c^2n^2.$$

Changing this to rectangular co-ordinates, we get

$$\epsilon^4 = a^2x^2 + b^2y^2 + c^2z^2.$$

It follows, therefore, that the principal diameters of the ellipsoid of gyration are the principal axes of the body, and the moment of inertia about the perpendicular on any tangent-plane is measured by the square of that perpendicular.

13. In all the succeeding chapters the moments of inertia of several bodies will be so frequently occurring that it will be found advisable to commit to memory the following table:

The moment of inertia of

(1) *A straight line of length* 2a

$$\left.\begin{array}{r}\text{about an axis through its centre}\\ \text{perpendicular to it}\end{array}\right\} = \text{mass}\,\frac{a^2}{3}.$$

(2) *A triangle*

$$\left.\begin{array}{r}\text{about an axis in its plane}\\ \text{through an angular point}\end{array}\right\} = \text{mass}\,\frac{1}{6}\,\frac{\gamma^3-\beta^3}{\gamma-\beta},$$

where β, γ are the distances of the other two angular points from the axis.

(3) *An ellipse semi-axes* a *and* b

$$\text{about the major axis } a = \text{mass}\,\frac{b^2}{4},$$

$$\text{.........} \quad \text{minor axis } b = \text{mass}\,\frac{a^2}{4};$$

$$\left.\begin{array}{r}\text{about an axis perpendicular to}\\ \text{its plane through the centre}\end{array}\right\} = \text{mass}\,\frac{a^2+b^2}{4}.$$

This includes the case of a circle.

(4) *An ellipsoid semi-axes* a, b, *and* c

$$\text{about the axis } a = \text{mass}\,\frac{b^2+c^2}{5}.$$

This includes the case of a sphere.

(5) *A cube whose side is* a

$$\left.\begin{array}{r}\text{about any axis through its}\\ \text{centre of gravity}\end{array}\right\} = \text{mass}\,\frac{a^2}{6}.$$

All these moments of inertia, except that of the triangle, are about principal axes at the centre of gravity. Then by

Prop. II. we can determine the moment of inertia about any other line through the centre of gravity, and by Prop. I. the moment about any parallel line.

14. As the process for determining these moments of inertia is very nearly the same for all these cases, it will be sufficient to consider only two instances.

To determine the moment of inertia of an ellipse about the minor axis.

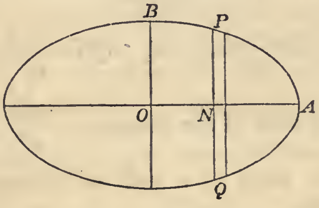
Let the equation to the ellipse be

$$y = \frac{b}{a}\sqrt{a^2 - x^2}.$$

Take any elementary area PQ parallel to the axis of y, then clearly the moment of inertia is

$$4\mu\int_0^a x^2 y\,dx = 4\mu\frac{b}{a}\int_0^a x^2\sqrt{a^2 - x^2}\,dx,$$

where μ is the mass of a unit of area.



To integrate this, put $x = a\sin\phi$, then the integral becomes

$$a^4\int_0^{\frac{\pi}{2}} \cos^2\phi\sin^2\phi\,d\phi = a^4\int_0^{\frac{\pi}{2}} \frac{1-\cos 4\phi}{8}\,d\phi = \frac{\pi a^4}{16};$$

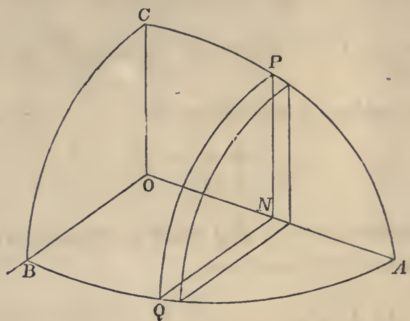
$$\therefore \text{ the moment of inertia} = \mu\pi ab\,\frac{a^2}{4} = \text{mass}\,\frac{a^2}{4}.$$

To determine the moment of inertia of an ellipsoid about a principal diameter.

Let the equation to the ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Take any elementary area PNQ parallel to the plane of yz. Its area is evidently $\pi PN \, . \, QN$. Now PN is the value of z



when $y=0$, and QN the value of y when $z=0$, as obtained from the equation to the ellipsoid;

$$\therefore \; PN = \frac{c}{a}\sqrt{a^2 - x^2}, \quad QN = \frac{b}{a}\sqrt{a^2 - x^2};$$

$$\therefore \text{ the area of the element} = \frac{\pi \, bc}{a^2}(a^2 - x^2).$$

Let μ be the mass of a unit of volume, then the whole moment of inertia

$$= \mu \int_{-a}^{a} \frac{\pi bc}{a^2}(a^2 - x^2) \frac{PN^2 + QN^2}{4} dx$$

$$= \mu \frac{\pi}{4} \frac{bc}{a^2} \int_{-a}^{a} (a^2 - x^2) \frac{b^2 + c^2}{a^2} (a^2 - x^2) \, dx$$

$$= \mu \frac{4}{3} \pi abc \frac{b^2 + c^2}{5}$$

$$= \text{mass} \frac{b^2 + c^2}{5}.$$

SECT. II. *On the Positions of the Principal Axes of a System.*

15. PROP. IV. *A straight line being given, it is required to find at what point in its length it is a principal axis of the system, and if any such point exist to find the other two principal axes at that point.*

Take the straight line as axis of z, and any point O in it as origin. Let C be the point at which it is a principal axis, and let Cx', Cy' be the other two principal axes.

Let $CO = h$, $\theta =$ angle between Cx' and Ox. Then

$$\left.\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \\ z' &= z - h \end{aligned}\right\}.$$

Hence

$$\left.\begin{aligned} \Sigma mx'z' = \cos\theta\,\Sigma mxz + \sin\theta\,\Sigma myz \\ -h(\cos\theta\,\Sigma mx + \sin\theta\,\Sigma my) \end{aligned}\right\} = 0 \;\ldots\ldots\; (1)$$

$$\left.\begin{aligned} \Sigma my'z' = -\sin\theta\,\Sigma mxz + \cos\theta\,\Sigma myz \\ -h(-\sin\theta\,\Sigma mx + \cos\theta\,\Sigma my) \end{aligned}\right\} = 0 \;\ldots\ldots\; (2)$$

$$\Sigma mx'y' = \Sigma m(y^2 - x^2)\frac{\sin 2\theta}{2} + \Sigma mxy\cos 2\theta = 0 \;\ldots\ldots\; (3)$$

The last equation shows that

$$\tan 2\theta = \frac{2\,\Sigma mxy}{\Sigma m(x^2 - y^2)} \;\ldots\ldots\ldots\ldots\; (4)$$

$$= \frac{2F}{B - A},$$

according to the previous notation.

The equations (1) and (2) must be satisfied by the same value of h. This gives as the condition that the axis of z is a principal axis at some point in its length,

$$\Sigma mxz\,\Sigma my = \Sigma myz\,\Sigma mx.$$

Substituting in (1), we get

$$h = \frac{\Sigma myz}{\Sigma my} = \frac{\Sigma mxz}{\Sigma mx} \quad \text{.......................} (5)$$

The equation (5) expresses the condition that the axis of z should be a principal axis at some point in its length; and the value of h gives the position of this point. The positions of the other two principal axes may then be found by equation (4).

To determine the geometrical meaning of this condition, take the plane of xz to pass through the centre of gravity of the body. Then we have $\Sigma my = 0$, and the equation becomes

$$\Sigma mx \, \Sigma myz = 0.$$

One of these factors must be zero. In order that h may be finite, we must have $\Sigma myz = 0$. Construct the momental ellipsoid at the centre of gravity. By Art. 7 its equation referred to axes of co-ordinates parallel to Cx, Cy, Cz, is

$$AX^2 + BY^2 + CZ^2 - 2EZX - 2FXY = \epsilon^4,$$

according to the previous notation. The coefficient D of YZ is zero, because by Art. 3

$$\Sigma myz = D + \Sigma m \,. \bar{y}\bar{z}.$$

The equation to a section parallel to the plane yz is

$$BY^2 + CZ^2 = \epsilon^4,$$

which is an ellipse referred to its principal diameters as axes. Hence, in order that a straight line may be a principal axis at some point not infinitely distant, it must be parallel to one of the principal diameters of the section of the momental ellipsoid at the centre of gravity, made by a plane perpendicular to the plane containing the axis and the centre of gravity.

16. If $\Sigma mxz = 0$ and $\Sigma myz = 0$, the equations (1) and (2) are both satisfied by $h = 0$. These are therefore the sufficient and necessary conditions that the axis of z should be a principal axis at the origin.

If the system be a plane lamina and the axis of z be a normal to the plane at any point, we have $z=0$. Hence the conditions $\Sigma mxz=0$ and $\Sigma myz=0$ are satisfied. Therefore one of the principal axes at any point of a lamina is a normal to the plane at that point.

In the case of a surface of revolution bounded by planes perpendicular to the axis, the axis is a principal axis at any point of its length.

17. Again equation (4) enables us, when one principal axis is given, to find the other two. If $\theta=\alpha$ be the first value of θ, all the others are included in $\theta=\alpha+n\frac{\pi}{2}$; hence all these values give only the same axes over again.

18. Since (4) does not contain h, it appears that if the axis of z be a principal axis at more than one point, the principal axes at those points are parallel. Again, in that case (5) must be satisfied by more than one value of h. But since h enters only in the first power, this cannot be unless

$$\Sigma mx=0, \quad \Sigma my=0,$$

$$\Sigma mxz=0, \quad \Sigma myz=0;$$

so that the axis must pass through the centre of gravity and be a principal axis at the origin, and therefore (since the origin is arbitrary) a principal axis at every point in its length.

If the principal axes at the centre of gravity be taken as the axes of x, y, z, (1) and (2) are satisfied for all values of h. Hence, if a straight line be a principal axis at the centre of gravity, it is a principal axis at every point in its length.

In many Dynamical investigations it is necessary to know the positions of the principal axes at any point of a body, and for this purpose the following rules will be useful.

19. Prop. V. *Given the positions of the principal axes* Ox, Oy, Oz *at the centre of gravity* O, *and the moments of inertia about them, to find the positions of the principal axes at any point* P *in the plane of* xy, *and the moments of inertia about those axes.*

Let the mass of the body be taken as unity, and let A, B be the moments of inertia about the axes Ox, Oy, of which we shall suppose A the greater.

Take two points S and H in the axis of x on each side of the origin so that

$$OS = OH = \sqrt{A - B}.$$

Then because these points are in one of the principal axes at the centre of gravity, the principal axes at S and H are parallel to the axes of co-ordinates, and the moments of inertia about those in the plane of xy are respectively A and $B + \overline{OS}|^2 = A$, and these being equal, any straight line through S and H in the plane of xy is a principal axis at that point, and the moment of inertia about it is equal to A.

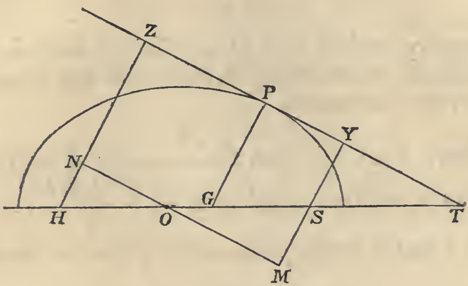
If P be any point in the plane of xy, then one of the principal axes at P will be perpendicular to the plane xy. For if p, q be the co-ordinates of P, the conditions that this line is a principal axis are

$$\left.\begin{array}{l}\Sigma m\,(x-p)\,z = 0\\ \Sigma m\,(y-q)\,z = 0\end{array}\right\},$$

which are obviously satisfied because the centre of gravity is the origin, and the principal axes the axes of co-ordinates.

The other two principal axes may be found thus. If two straight lines meeting at a point P be such that the moments of inertia about them are equal, then provided they are in a principal plane the principal axes at P bisect the angles between these two straight lines. For if with centre P we describe the Momental Ellipse, then the axes of this ellipse bisect the angles between any two equal radii vectores.

Join SP and HP; the moments of inertia about SP, HP are each equal to A. Hence, if PG and PT are the internal



and external bisectors of the angle SPH; PG, PT are the principal axes at P. *If therefore with* S *and* H *as foci we describe any ellipse or hyperbola, the tangent and normal at any point are the principal axes at that point.*

20. Take any straight line MN through the origin, making an angle θ with the axis of x. Draw SM, HN perpendiculars on MN. The moment of inertia about it is

$$\begin{aligned} &= A\cos^2\theta + B\sin^2\theta \\ &= A - (A-B)\sin^2\theta \\ &= A - \overline{OS\sin\theta}|^2 \\ &= A - SM^2. \end{aligned}$$

Through P draw PT parallel to MN, and let SY and HZ be the perpendiculars from S and H on it. The moment of inertia about PT is then

$$\begin{aligned} &= \text{moment about } MN + MY^2 \\ &= A + (MY - SM)(MY + SM) \\ &= A + SY.HZ. \end{aligned}$$

In the same way it may be proved that the moment of inertia about a line PG passing between H and S is *less* than A by the product of the perpendiculars from S and H on PG.

If therefore with S *and* H *as foci we describe any ellipse or hyperbola, the moments of inertia about any tangent to either of these curves is constant.*

It follows from this that the moments of inertia about the principal axes at P are equal to $B + \left(\frac{SP \pm HP}{2}\right)^2$.

For if a and b be the axes of the ellipse we have

$$a^2 - b^2 = OS^2 = A - B,$$

and hence

$$A + SY \, . \, HZ = A + b^2 = B + a^2 = B + \left(\frac{SP + HP}{2}\right)^2,$$

and the hyperbola may be treated in a similar manner.

21. Prop. VI. *Given the positions of the principal axes* Ox, Oy, Oz *at the centre of gravity* O, *and the moments of inertia* A, B, C *about them, to find the positions of the principal axes at any point.*

Let the mass of the body be taken as unity, and let p, q, r be the co-ordinates of P. About the point P describe the Momental Ellipsoid. Its equation referred to axes through P as origin parallel to Ox, Oy, Oz is known to be

$$A'X^2 + B'Y^2 + C'Z^2 - 2\,(\Sigma my'z')\,YZ - 2\,(\Sigma mz'x')\,ZX$$
$$- 2\,(\Sigma mx'y')\,XY = \epsilon^4 \ldots\ldots\ldots\ldots\ldots(1),$$

where x', y', z' are the co-ordinates of the particle m referred to axes through P parallel to the axes of co-ordinates. Now by Prop. I.

$$\left.\begin{array}{c} \Sigma my'z' = qr, \ \Sigma mz'x' = rp, \ \Sigma mx'y' = pq \\ A' = A + q^2 + r^2, \ B' = B + r^2 + p^2, \ C' = C + p^2 + q^2 \end{array}\right\} \ldots(2).$$

Hence the equation to the ellipsoid becomes

$$A'X^2 + B'Y^2 + C'Z^2 - 2qr\,YZ - 2rp\,ZX - 2pq\,XY = \epsilon^4 \ldots\ldots\ldots\ldots (3).$$

Suppose that this ellipsoid when referred to its principal axes is

$$I_1X^2 + I_2Y^2 + I_3Z^2 = \epsilon^4.$$

Then I_1, I_2, I_3 are the roots of the Discriminating Cubic

$$(I - A')(I - B')(I - C') - q^2r^2(I - A') - r^2p^2(I - B') - p^2q^2(I - C') + 2p^2q^2r^2 = 0 \ldots\ldots (4).$$

And if l, m, n be proportional to the direction-cosines of the axis corresponding to any one of the values of I, their values may be found from

$$\left.\begin{array}{l} A'l - pq.m - pr.n = I.l \\ B'm - qp.l - qr.n = I.m \\ C'n - rp.l - rq.m = I.n \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Now the centre of gravity being the origin, consider the ellipsoid whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

and suppose the point p, q, r to be situated on this ellipsoid, hence

$$\frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

The normal to this ellipsoid at the point p, q, r may be made to coincide with one of the principal axes at that point.

For the direction-cosines of the normal are respectively proportional to

$$l = \frac{p}{a^2}, \quad m = \frac{q}{b^2}, \quad n = \frac{r}{c^2}.$$

Substituting these in equations (5), and remembering equation (6) we get

$$I_1 = A' - a^2 + p^2,$$
$$= A + p^2 + q^2 + r^2 - a^2, \text{ by (2)}.$$

Similarly $I_1 = B + p^2 + q^2 + r^2 - b^2,$

$$I_1 = C + p^2 + q^2 + r^2 - c^2,$$

from the second and third. That these equations may coexist, we must have

$$a^2 - A = b^2 - B = c^2 - C \dots\dots\dots\dots\dots (7),$$

or the ellipsoid must be confocal with that whose equation is

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = 1.$$

Again, equation (4) will be satisfied by this value of I, for it may be deduced from (5) by elimination of l, m, n.

Similarly it may be shown that the other two principal axes at P are normals to the other two confocal surfaces of the second order*.

As we shall have to make frequent reference to these surfaces we shall call them the "subsidiary" surfaces of the second order. Calling each of the quantities in (7) λ, their general equation will be

$$\frac{x^2}{A+\lambda} + \frac{y^2}{B+\lambda} + \frac{z^2}{C+\lambda} = 1,$$

and λ is the variable parameter that determines the particular ellipsoid under consideration.

The ellipsoid that corresponds to $\lambda = 0$ is evidently the ellipsoid of gyration.

22. We know that the lines of curvature on an ellipsoid are the curves in which it is intersected by confocal surfaces

* This result was given by Prof. Thomson in the *Cambridge and Dublin Mathematical Journal,* 1846.

of the second order, and that the tangents to the lines of curvature meeting in any point are parallel to the axes of the diametral section made by a plane parallel to the tangent plane at that point.

Hence if through any point we describe the subsidiary ellipsoid, the principal axes are the normal to the ellipsoid at that point, and the tangents to the lines of curvature, or which is the same thing, parallels to the axes of the section parallel to the tangent plane.

23. PROP. VII. *To find the moments of inertia about the three principal axes meeting at a point whose co-ordinates are p, q, r.*

The values of I given by equations (5) are the squares of the reciprocals of the axes of the momental ellipsoid at the point P. Hence the values of I given by (5) are the required moments of inertia. Let P be the length of the radius vector drawn from the centre of the "subsidiary" ellipsoid to the point (p, q, r). Let Q and R be the lengths of the axes of the diametral plane of the radius vector P. The equation to this diametral plane is

$$\frac{px}{a^2}+\frac{qy}{b^2}+\frac{rz}{c^2}=0,$$

and the direction-cosines of one of the axes of this section are proportional to

$$\frac{p}{Q^2-a^2},\ \frac{q}{Q^2-b^2},\ \frac{r}{Q^2-c^2}.$$

Hence substituting in (5)

$$\begin{aligned}I_2 &= A' - q^2\frac{Q^2-a^2}{Q^2-b^2} - r^2\frac{Q^2-a^2}{Q^2-c^2}\\ &= A + p^2 + q^2 + r^2\\ &\quad - (Q^2-a^2)\left\{\frac{p^2}{Q^2-a^2}+\frac{q^2}{Q^2-b^2}+\frac{r^2}{Q^2-c^2}\right\}.\end{aligned}$$

Now the equation for finding the axes of the section is

$$\frac{p^2}{Q^2-a^2}+\frac{q^2}{Q^2-b^2}+\frac{r^2}{Q^2-c^2}=0,$$

multiply this equation by Q^2, and subtract from it

$$\frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2}=1\,;$$

and we get

$$\frac{p^2}{Q^2-a^2}+\frac{q^2}{Q^2-b^2}+\frac{r^2}{Q^2-c^2}=-1.$$

Hence $$I_2=A+p^2+q^2+r^2+Q^2-a^2$$

$$=P^2+Q^2-\lambda.$$

Similarly $$I_3=P^2+R^2-\lambda.$$

And the value of I_1, the moment of inertia about the normal, has been already shown to be

$$I_1=P^2-\lambda.$$

We can deduce from these equations

$$I_1+I_2+I_3=A+B+C+2P^2,$$

$$I_2+I_3-I_1=A+B+C+2\lambda.$$

The last result is constant for all points on the subsidiary ellipsoid.

24. Prop. VIII. *To find the curves on the subsidiary ellipsoid through a given point at which the principal moments of inertia are equal to those at the given point.*

First, all those points at which the principal moment about the normal is constant, are found by making I_1 constant. This gives P constant, and hence the required curve is the intersection of a sphere with the ellipsoid. This curve is well known to be the spherical ellipse.

Secondly, those points at which the principal moment about a tangent is constant are found by making I_2 or I_3 constant.

Let Π be the length of the perpendicular from the origin on the plane that contains the axes I_2, I_3, then

$$Q \,.\, R \,.\, \Pi = abc.$$

Also $$P^2 + Q^2 + R^2 = a^2 + b^2 + c^2.$$

Hence $$I_2 = P^2 + Q^2 + R^2 - \lambda - R^2$$

$$= a^2 + b^2 + c^2 - \lambda - \frac{a^2b^2c^2}{Q^2\Pi^2},$$

and $$I_3 = a^2 + b^2 + c^2 - \lambda - \frac{a^2b^2c^2}{R^2\Pi^2}.$$

If any line of curvature be taken on the ellipsoid the tangents are all principal axes at the points of contact, and (as is proved in all books on Solid Geometry) along that line

$$Q \,.\, \Pi = \text{constant}.$$

Hence the moments of inertia about the tangents to any line of curvature are equal.

In the plane that contains the axes I_2, I_3, and through their point of intersection draw any straight line making an angle ϕ with the axis I_2. Then the moment of inertia about this line is

$$\begin{aligned} I &= I_2 \cos^2\phi + I_3 \sin^2\phi \\ &= P^2 - \lambda + Q^2 \cos^2\phi + R^2 \sin^2\phi \\ &= P^2 + Q^2 + R^2 - \lambda - Q^2R^2 \left(\frac{\cos^2\phi}{Q^2} + \frac{\sin^2\phi}{R^2}\right) \\ &= a^2 + b^2 + c^2 - \lambda - \frac{a^2b^2c^2}{\Pi^3} \cdot \frac{1}{D^2}, \end{aligned}$$

where D is the radius vector, parallel to the axis I, of the section of the ellipsoid made by a plane drawn through the centre parallel to the tangent plane at the point P.

But if any geodesic line be drawn on an ellipsoid, and if D be the semi-diameter of the surface parallel to any tangent to the geodesic line, and Π the perpendicular on the tangent-plane to the ellipsoid at the point of contact of the tangent,

$$\Pi D = \text{constant}.$$

Hence the moments of inertia about the tangents to any geodesic line on the ellipsoid are equal.

25. The locus of all those points at which one of the principal moments of inertia of the body is constant is called an equimomental surface.

To find the equation to such a surface we have only to put I_1 constant, this gives $\lambda = P^2 - I$. Hence the equation to the surface becomes

$$\frac{x^2}{x^2+y^2+z^2+A-I} + \frac{y^2}{x^2+y^2+z^2+B-I} + \frac{z^2}{x^2+y^2+z^2+C-I} = 1.$$

26. Prob. *To find the locus of all those points in a body at which the product of the three principal moments is equal to a given quantity.*

Take the principal axes at the centre of gravity for axes of co-ordinates, and let A, B, C, be the moments of inertia about them. Let x, y, z, be any point of the required locus. Then if we construct the momental ellipsoid whose centre is at (x, y, z), the product of its three axes is equal to a given quantity. The equation to this ellipsoid is

$$A'X^2 + B'Y^2 + C'Z^2 - 2yzYZ - 2zxZX - 2xyXY = \epsilon^4,$$

where $A' = A + y^2 + z^2$, with similar expressions for B' and C'.

The axes of this surface may be found from the Discriminating Cubic

$$(P-A')(P-B')(P-C')-y^2z^2(P-A')-z^2x^2(P-B')$$
$$-x^2y^2(P-C')+2x^2y^2z^2=0.$$

The term independent of P in this equation is equal to the product of the three principal moments, and is therefore to be made constant. Hence, the equation to the required locus is

$$-A'B'C'+y^2z^2A'+z^2x^2B'+x^2y^2C'+2x^2y^2z^2=Q,$$

where Q is the constant product of the three principal moments.

EXAMPLES.

1. Find the moment of inertia of an arc of a circle whose radius is a, and which subtends an angle 2α at the centre:

(1) About an axis through its centre perpendicular to its plane. *Result.* Ma^2.

(2) About an axis through its middle point perpendicular to its plane. *Result.* $2M\left(1-\dfrac{\sin\alpha}{\alpha}\right)a^2$.

(3) About the diameter which bisects the arc.
Result. $M\left(1-\dfrac{\sin 2\alpha}{2\alpha}\right)\dfrac{a^2}{2}$.

(4) About its chord.

2. The moment of inertia of a right cone of mass M about the axis is $M\dfrac{3}{10}b^2$. That about a straight line through

the vertex perpendicular to the axis is $M\frac{3}{5}\left(a^2+\frac{b^2}{4}\right)$, and about a slant side $M\frac{3}{20}\;\frac{6a^2+b^2}{a^2+b^2}b^2$, where a is the altitude of the cone and b the radius of the base.

3. The moment of inertia of an arc of an equiangular spiral measured from the pole to the extremity of the radius vector r is $M\frac{1}{3}r^2$ about an axis perpendicular to its plane through the pole.

4. The moment of inertia of the part of a parabola cut off by any ordinate at a distance x from the vertex is $M\frac{3}{7}x^2$ about the tangents at the vertex, and $M\frac{y^2}{5}$ about the principal diameter, where y is the ordinate corresponding to x.

5. The moment of inertia of the lemniscate $r^2=a^2\cos 2\theta$ about a line through the origin, in its plane and perpendicular to its axis is

$$M\frac{3\pi+8}{48}a^2.$$

6. The moment of inertia of a triangle about a line perpendicular to its plane through its centre of gravity is

$$\frac{M}{36}(a^2+b^2+c^2),$$

where a, b, c are the sides of the triangle.

7. The surfaces of equal density in a heterogeneous body are a family of closed similar and similarly situated surfaces, having given the moment of inertia of a homogeneous body bounded by any one of these surfaces, find that of the heterogeneous body about the same axis.

Let $D\,\phi(\alpha)$ be the moment of inertia of the homogeneous body of density D, bounded by the surface whose parameter is α, and let ρ be the density of the heterogeneous body along this surface. Then $\int_0^a \rho\, \phi'(a)\, da$ is the required moment of inertia.

Apply this method to find the moment of inertia about the major axis of an ellipsoid whose strata of equal density are similar concentric ellipsoids, the density along the major axis varying as the distance from the centre.

Result. $M\frac{2}{9}(b^2 + c^2)$.

8. The angle of an equiangular spiral is $\cot^{-1}\frac{2}{3}$, prove that the principal axes of an arc subtending an angle $\frac{2n+1}{2}\pi$ at the pole with respect to the pole, are inclined at an angle $\frac{\pi}{8}$ to the extreme radii vectores.

9. The principal axes of a right-angled triangle are one perpendicular to the plane, and two others inclined to its sides at the angle

$$\frac{1}{2}\tan^{-1}\frac{ab}{a^2 - b^2},$$

where a and b are the sides of the triangle adjacent to the right angle.

10. Find the positions of the principal axes of a cube at any given point.

11. Two particles, each of mass $= m$, are placed at the extremities of the minor axis of an elliptic area of mass M. Prove that two of the principal axes at any point of the circumference of the ellipse will be the tangent and normal to the ellipse, provided $\frac{m}{M} = \frac{3}{8}\frac{e^2}{1-2e^2}$.

12. If k_1, k_2 be the radii of gyration of an elliptic lamina about two conjugate diameters $\frac{1}{k_1^2} + \frac{1}{k_2^2}$ is constant.

13. The moment of inertia of an elliptic area about any diameter is proportional inversely to the whole length of the focal chord parallel to that diameter.

14. Find the locus of those diameters of an ellipsoid, the moments of inertia about which are equal to the moment of inertia about the mean axis.

15. Determine the conditions that there may be a point such that the moment of inertia about every axis through that point is the same.

Result. Two of the principal moments at the centre of gravity must be equal, and each must be less than the third principal moment. There are then two points in the axis of unequal moment which satisfy the conditions.

16. If A', B', C' be the moments of inertia of a body about any three straight lines at right angles meeting in a point O, and if A, B, C be the moments of inertia about the three principal axes at O, prove that

$$A' + B' + C' = A + B + C,$$

and that each of these quantities is equal to Σmr^2, where r is the distance of any particle m of the body from O.

17. If two principal moments of inertia of a body be equal, the equation to a curve such that any tangent is a principal axis at the point of contact is of the form

$$\theta = A + Br + Cr^2 + Dr^3 + Er^4,$$

its pole being the centre of gravity, and its plane passing through the principal axis of unequal moment.

18. If A', B', C' be the moments of inertia about principal axes through a point P, A, B, C those about principal axes through the centre of gravity; prove that (1) when $A' + B' - C' = A + B - C$ the locus of P is one of the principal planes through the centre of gravity; (2) when

$$(\sqrt{A'} + \sqrt{B'} + \sqrt{C'})\,(\sqrt{A'} + \sqrt{B'} - \sqrt{C'})\,(\sqrt{B'} + \sqrt{C'} - \sqrt{A'})\\(\sqrt{C'} + \sqrt{A'} - \sqrt{B'})$$

is constant, the locus is an ellipsoid similar, similarly situated, and concentric with the central ellipsoid at the centre of gravity; (3) when $B = C$ and each is less than A, and P lies on a lemniscate of revolution, having for foci the points where the central ellipsoid is a sphere, $A' - B' = A - B$, A and B being the moments about the axes through P which pass through the axis A.

19. The particles of a body attract an external point according to the law of nature. Prove that the resultant attraction on every external point can be the same as that of a mass M collected at a fixed point O only when (1) M is equal to the mass of the attracting body, and (2) every axis of the body at O is a principal axis.

CHAPTER II.

D'ALEMBERT'S PRINCIPLE, &c.

27. A RIGID body is a collection of material particles connected together by invariable geometrical relations. Our first attempt therefore to determine the motion of such a body would be to write down the equations of the several particles according to the principles laid down in treatises on Dynamics of a particle, and then to eliminate the unknown reactions between the particles, and thus obtain the equations of motion of a rigid body.

But if we attempt to do this, we are at once stopped by our ignorance of the nature of the actions of one particle on another. It would be necessary to make some assumptions in regard to these.

We might assume *first*, that the action between two particles is along the line which joins them; *secondly*, that the action and reaction between any two are equal and opposite.

The equations of motion of each separate particle on these assumptions may be easily written down. Suppose there are n particles, then there will be $3n$ equations, and as shown in Todhunter's *Statics*, Chap. VI. there must be at least $3n-6$ unknown reactions. It is therefore clear that after the elimination has been effected there cannot be *more than* 6 resulting equations free from the unknown reactions.

But if the equations are written down it will be seen that the reactions enter into the equation in such a manner that we can always eliminate them, however numerous the particles may be, and obtain six resulting equations.

But D'Alembert proposed a method by which these six resulting equations may be obtained without writing down the equations of motion of the several particles, and without making any assumption as to the nature of their mutual actions except the following:—

"The Internal actions and reactions of any rigid system in motion are in equilibrium amongst themselves."

28. PROP. *To explain D'Alembert's Principle, and to obtain the equations of motion of a rigid System.*

Consider any particle of a system whose mass is m. Let F be the resultant of the impressed forces, R the resultant of the internal actions on m. Let f be the resultant acceleration of the particle, which may be called the effective accelerating force on m.

Then by Dynamics of a particle, mf is the resultant of F and R; hence if we reverse the effective force on any particle, the three forces F, R, mf are in equilibrium. If we apply the same reasoning to every particle of the system, it is evident, that the whole group of forces mf will be in equilibrium with the groups F and R.

Now D'Alembert's principle asserts that the group of forces R will itself be in equilibrium; whence it follows that the group of forces F will be in equilibrium with the group mf.

Hence, *if forces equal to the effective forces but acting in exactly opposite directions were applied at each point of the system, these would be in equilibrium with the impressed forces.*

29. This proposition is usually called D'Alembert's principle, and we may at once deduce from it the equations of motion.

Let x, y, z be the co-ordinates of the particle m at the time t referred to any set of rectangular axes fixed in space.

Then $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$ will be the effective accelerations of the particle.

Let X, Y, Z be the impressed accelerating forces on the same particle resolved parallel to the axes.

By D'Alembert's principle the forces

$$m\left(X-\frac{d^2x}{dt^2}\right),\quad m\left(Y-\frac{d^2y}{dt^2}\right),\quad m\left(Z-\frac{d^2z}{dt^2}\right),$$

together with the similar forces on every particle, will be in equilibrium. Hence, by the principles of Statics, we have the six general equations

$$\left.\begin{aligned}\Sigma\left(m\frac{d^2x}{dt^2}\right)&=\Sigma(mX)\\ \Sigma\left(m\frac{d^2y}{dt^2}\right)&=\Sigma(mY)\\ \Sigma\left(m\frac{d^2z}{dt^2}\right)&=\Sigma(mZ)\end{aligned}\right\}\text{......................(A).}$$

These are obtained by resolving parallel to the axes.

$$\left.\begin{aligned}\Sigma m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right)&=\Sigma m(xY-yX)\\ \Sigma m\left(y\frac{d^2z}{dt^2}-z\frac{d^2y}{dt^2}\right)&=\Sigma m(yZ-zY)\\ \Sigma m\left(z\frac{d^2x}{dt^2}-x\frac{d^2x}{dt^2}\right)&=\Sigma m(zX-xZ)\end{aligned}\right\}\text{......(B).}$$

These are obtained by taking moments about the axes.

In a precisely similar manner by taking the expressions for the accelerations in Polar Co-ordinates, we should have obtained another, but equivalent, set of equations of motion.

30. These six equations together with the geometrical relations are sufficient for the determination of the motion of any

system; for the motion of a rigid body is known when the motions of any three points not in one straight line are known. By Geometry we can determine the co-ordinates of every other particle of a rigid body in terms of the co-ordinates of these three chosen points; substituting in the equations of motion we have *six* equations between the co-ordinates of the three points. Joining these to the *three* geometrical equations expressing the fact that the distance between any two of the three points remains invariable, we have *nine* equations to determine the nine co-ordinates of the three chosen points. If there be more than one body in the system, the same thing will be true for each body. If there be any unknown reactions it is obvious that the very circumstance which causes the reaction will give an additional geometrical equation.

31. In D'Alembert's principle no assumption has been made as to the *nature* of the actions between the particles; hence the principle is true whether the particles be rigidly connected or not. We may, for example, apply the principle to the case of a fluid in motion or to any elastic or flexible body.

The principle is in reality an extension of the first law of motion. That law is equivalent to an assertion that the molecular actions of the particles which constitute a body do not affect the motion of translation of that body. D'Alembert's principle asserts further that they do not affect its motion when that motion consists of a combination of a motion of translation with one of rotation.

The truth of the principle cannot be established by abstract reasoning. It must be considered as resting on experimental evidence, or rather on that *inductive proof* which is derived from the accurate coincidence of the results of calculations founded on this principle with the observed motions of a rigid body.

32. We have seen that the six general equations of motion are sufficient for the solution of every Dynamical Problem, but in their present form they are too complicated to be of much use. Certain general principles have therefore

been deduced from them which will greatly facilitate their application to any particular case. The two most important of these are:

First. The motion of the centre of gravity of a system acted on by any forces is the same as if all the mass were collected at the centre of gravity, and all the forces were applied at that point parallel to their former directions.

Secondly. The motion of a body acted on by any forces, about its centre of gravity, is the same as if the centre of gravity were fixed, and the same forces acted on the body.

These are called the "Principles of the Conservation of the Motions of translation and rotation."

The first principle follows from the set (A) of the general equations of motion.

Taking any one of them we have

$$\Sigma\left(m\frac{d^2x}{dt^2}\right) = \Sigma\,(mX).$$

Let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the co-ordinates of the centre of gravity, then

$$\bar{x}\,.\,\Sigma m = \Sigma mx\,;$$

$$\therefore\ \frac{d^2\bar{x}}{dt^2}\Sigma m = \Sigma m\frac{d^2x}{dt^2}$$

$$= \Sigma mX \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and the other two equations may be treated in a similar manner.

But these are the equations which give the motions of a mass Σm acted on by forces $\Sigma\,(mX)$, &c. Hence the principle follows.

The second principle may be deduced from the set (B) of the general equations of motion.

Taking any one of them we have

$$\Sigma m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right)=\Sigma m\,(xY-yX)\ldots\ldots(2).$$

Let $\qquad x=\bar{x}+x',\; y=\bar{y}+y',\; z=\bar{z}+z'.$

Substituting, the equation becomes

$$\Sigma m\left\{(\bar{x}+x')\left(\frac{d^2\bar{y}}{dt^2}+\frac{d^2y'}{dt^2}\right)-(\bar{y}+y')\left(\frac{d^2\bar{x}}{dt^2}+\frac{d^2x'}{d^2t}\right)\right\}$$
$$=\Sigma m\,\{(\bar{x}+x')\,Y-(\bar{y}+y')\,X\}.$$

Expanding*, we get

$$\left(\bar{x}\frac{d^2\bar{y}}{dt^2}-\bar{y}\frac{d^2\bar{x}}{dt^2}\right)\Sigma m+\Sigma m\left(x'\frac{d^2y'}{dt^2}-y'\frac{d^2x'}{dt^2}\right)$$
$$+\bar{x}\,\Sigma m\,\frac{d^2y'}{dt^2}-\bar{y}\,\Sigma m\,\frac{d^2x'}{dt^2}+\frac{d^2\bar{y}}{dt^2}\,\Sigma m\,x'-\frac{d^2\bar{x}}{dt^2}\,\Sigma m\,y'$$
$$=\bar{x}\,\Sigma m\,Y-\bar{y}\,\Sigma m\,X+(\Sigma m\,x'Y-y'X)\ldots\ldots\ldots\ldots\ldots(3).$$

But $\qquad \Sigma m\,x'=0,$ and $\Sigma m\,y'=0;$

$$\therefore\ \Sigma m\frac{d^2x'}{dt^2}=0,\ \text{and}\ \Sigma m\frac{d^2y'}{dt^2}=0.$$

* This demonstration may be much shortened by the consideration that the origin of co-ordinates is quite arbitrary. Let it be so chosen that the centre of gravity is passing through it at the moment under consideration. Then $\bar{x}=0$, $\bar{y}=0$, and the equation becomes

$$\Sigma m\left(x'\frac{d^2y'}{dt^2}-y'\frac{d^2x'}{dt^2}\right)+\frac{d^2\bar{y}}{dt^2}\Sigma mx'-\frac{d^2\bar{x}}{dt^2}\Sigma my'$$
$$=\Sigma m(x\,Y-y'X).$$

But since $\Sigma mx'=0$, $\Sigma my'=0$, this at once reduces to the equation (4) in the text.

It is obvious that the above transformation of the equation (2) might be deduced from the general proposition in Art. 5.

Hence the whole of the second line of the left-hand side of equation (3) vanishes. Again,

$$\frac{d^2\bar{x}}{dt^2}\Sigma m = \Sigma m X, \text{ and } \frac{d^2\bar{y}}{dt^2}\Sigma m = \Sigma m Y.$$

Hence the above equation reduces to

$$\Sigma m\left(x'\frac{d^2y'}{dt^2} - y'\frac{d^2x'}{dt^2}\right) = \Sigma m\,(x'Y - y'X)\;\ldots\ldots(4),$$

and the other two equations may be treated in a similar way.

But these are exactly the equations we should have obtained if we had regarded the centre of gravity as a fixed point and taken it as the origin of moments. Hence the principle follows.

33. From the two General Principles we may deduce the following Corollaries.

First. If there be a system of bodies subject only to their mutual actions, their centre of gravity is either at rest or moves uniformly in a straight line.

By the term "mutual action" is meant any action of one body of the system on another which is balanced by the equal and opposite reaction of that other, such as their mutual attractions or the tensions of any strings elastic or inelastic joining two of the bodies.

The centre of gravity of the solar system, for example, is either at rest or moves uniformly in a straight line; the fixed stars being supposed too distant to exert any perceptible attraction.

Secondly. If a body be acted on by any number of forces which are statically equivalent to a single force acting at each instant through the centre of gravity of the body, then the motion of the body about its centre of gravity will be undisturbed by the action of these forces.

Thus if we suppose the earth, planets, sun, &c. to be spherical bodies such that the density of each varies as any function of the distance from the centre of that body, then the attraction of any one on any other would be the same as if each were collected into one particle at its respective centre of gravity. In this case therefore the rotation of each heavenly body is unaffected by the attraction of all the others.

EXAMPLES.

1. Two particles moving in the same plane are projected in parallel but opposite directions with velocities inversely proportional to their masses. Find the motion of their centre of gravity.

2. A person is placed on a perfectly smooth table, show how he may get off.

3. A person is placed at one end of a perfectly rough board which rests on a smooth table. Supposing he walks to the other end of the board, determine how much the board has moved. Supposing that he stepped off the board, show how to determine its subsequent motion.

4. The motion of the centre of gravity of a shell shot from a gun in vacuo is a parabola, and its motion is unaffected by the bursting of the shell.

5. A rod revolving uniformly in a horizontal plane round a pivot at its extremity suddenly snaps in two: determine the motion of each part.

6. A uniform chain of length c is held so that one extremity just touches an inelastic plane, and is only under the action of a force in its length produced at a distance a on the opposite side of the plane; show that the last particle of the chain when let go will strike the plane with a velocity $\sqrt{\frac{2\mu}{a} \cdot \log \frac{c+a}{a}}$, μ being the absolute intensity of the force which varies inversely as the square of the distance.

CHAPTER III.

MOTION ABOUT A FIXED AXIS.

34. PROP. *To determine the motion of a body about a fixed axis under the action of any forces.*

Let any plane passing through the axis and fixed in space be taken as a plane of reference, and let θ be the angle which any other plane through the axis and fixed in the body, makes with the first plane. Then our object is to determine θ as a function of t.

Take any element m of the body and let its radius vector r perpendicular to the axis make an angle ϕ with the plane of reference. The effective moving forces on this element are $mr\frac{d^2\phi}{dt^2}$ and $-mr\left.\overline{\frac{d\phi}{dt}}\right|^2$, perpendicular to and along the direction of r. If these, taken throughout the system, be reversed, they will be in equilibrium with the impressed forces and with the reactions at the axis.

Taking moments about the axis we have

$$\Sigma m\left(r^2\frac{d^2\phi}{dt^2}\right) = L,$$

where L is the moment of the impressed forces about the axis.

Now since the particles of the body are rigidly connected with each other, it is obvious that $\frac{d^2\phi}{dt^2}$ is the same for every particle and equal to $\frac{d^2\theta}{dt^2}$. Hence

$$\frac{d^2\theta}{dt^2}\,.\,\Sigma mr^2 = L,$$

or
$$\frac{d^2\theta}{dt^2} = \frac{\text{moment of forces about axis}}{\text{moment of inertia about axis}}.$$

This equation when integrated will give the values of θ and $\frac{d\theta}{dt}$ at any time, and thus the position and velocity of the body will have been found.

35. PROP. *A body moves about a fixed horizontal axis acted on by gravity only, to determine the motion.*

Take the vertical plane through the axis as the plane of reference, and the plane through the axis and the centre of gravity as the plane fixed in the body. Then the equation of motion is

$$\frac{d^2\theta}{dt^2} = \frac{\text{moment of forces}}{\text{moment of inertia}} \quad \text{.........................} \quad (1)$$

$$= -\frac{Mgh \sin\theta}{M(k^2+h^2)},$$

where h is the distance of the centre of gravity from the axis and Mk^2 is the moment of inertia of the body about an axis through the centre of gravity parallel to the fixed axis.

Hence
$$\frac{d^2\theta}{dt^2} + \frac{gh}{k^2+h^2}\sin\theta = 0 \quad \text{......................} \quad (2).$$

The equation (2) cannot be integrated in finite terms, but if the oscillations be small, we may reject the cubes and higher powers of θ and the equation will become

$$\frac{d^2\theta}{dt^2} + \frac{gh}{k^2+h^2}\theta = 0.$$

Hence the time of a complete oscillation is

$$2\pi\sqrt{\frac{k^2+h^2}{gh}}.$$

If h and k be measured in feet and $g = 32{\cdot}18$, this formula gives the time in seconds.

36. The equation of motion of a particle of any mass suspended by a string l is

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}.\sin\theta = 0 \quad\text{.....................}\quad (3),$$

which may be deduced from equation (2) by putting

$$k = 0 \text{ and } h = l.$$

Hence the angular motions of the string and the body under the same initial conditions will be identical if

$$l = \frac{k^2+h^2}{h} \quad\text{............................}(4).$$

This length is called the "length of the simple equivalent pendulum."

Through G, the centre of gravity of the body, draw a perpendicular to the axis of revolution cutting it in C. Then C is called the *centre of suspension*. Produce CG to O so that $CO = l$. Then O is called the *centre of oscillation*. If the whole mass of the body were collected at the centre of oscillation and suspended by a thread to the centre of suspension, its angular motion and time of oscillation would be the same as that of the body under the same initial circumstances.

37. The equation (4) may be put under another form. Since $CG = h$ and $OG = l - h$, we have

$$CG \,.\, OG = k^2.$$

This equation shows that if O be made the centre of suspension, the axis being parallel to the axis about which k was taken, then C will be the centre of oscillation. Thus *the centres of oscillation and suspension are convertible and the time of oscillation about each is the same.*

38. If the time of oscillation be given, l is given and the equation (4) will give two values of h. Let these values be h_1, h_2. Let two cylinders be described with that straight line as axis about which the radius of gyration k was taken, and let the radii of these cylinders be h_1, h_2. Then the times of oscillation of the body about any generating lines of these cylinders are the same, and are approximately equal to

$$2\pi\sqrt{\frac{l}{g}}\,.$$

39. To find the axis of suspension parallel to a fixed line in the body such that the time of oscillation is a minimum it will be sufficient to make l a minimum. Now

$$l = \frac{h^2 + k^2}{h}\,.$$

Differentiating, k being a constant as before, we have

$$0 = 1 - \frac{k^2}{h^2}\,;$$

$$\therefore\ h = k.$$

About G as centre describe a circle with radius k in the plane perpendicular to the fixed line. Then the time of oscillation about any axis through the circumference of this circle is less than if the axis had been taken nearer to or further off from the centre of gravity.

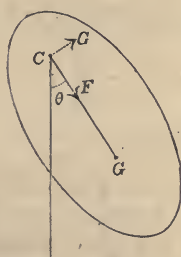
Since $h_1h_2=k^2$, we have in this case $h_1=h_2$, each being $=k$, and the two cylinders above mentioned become identical. In this case the length l of the simple equivalent pendulum is $=2k$.

The time of oscillation about the axis thus found is not an absolute minimum. It is a minimum for all axes drawn parallel to a given straight line in the body. To find the axis about which the time is absolutely a minimum we must find the axis about which k is a minimum. Now it is proved in Art. 11, that of all axes through G the axis about which the moment of inertia is least or greatest is one of the principal axes. Hence the axis about which the time of oscillation is a minimum is parallel to that principal axis through G about which the moment of inertia is least. And if Mk^2 be the moment of inertia about that axis, the axis of suspension is at a distance k measured in any direction from the principal axis.

PROP. *A body moves about a fixed axis under the action of any forces, to find the pressures on the axis.*

40. *First*, Suppose the body and the forces to be symmetrical about the plane through the centre of gravity perpendicular to the axis. Then it is evident that the pressures on the axis are reducible to a single force at C the centre of suspension.

Let F, G be the actions of the point of support on the



body resolved along and perpendicular to CG. Let X, Y be the sum of the resolved parts of the impressed forces in the same directions, and L their moment round C.

Let $CG = h$ and $\theta =$ angle which CG makes with any straight line fixed in space.

Taking moments about C, we have

$$\frac{d^2\theta}{dt^2} = \frac{L}{M(k^2 + h^2)} \quad\text{.....................}\quad (1).$$

The motion of the centre of gravity is the same as if all the forces acted at that point. Now it describes a circle round C; hence, taking the tangential and normal resolutions, we have

$$h\frac{d^2\theta}{dt^2} = \frac{Y+G}{M} \quad\text{.....................}\quad (2)$$

$$-h\left.\frac{d\theta}{dt}\right|^2 = \frac{X+F}{M} \quad\text{.....................}\quad (3).$$

Equation (1) gives the values of $\frac{d^2\theta}{dt^2}$ and $\frac{d\theta}{dt}$, and then the pressures may be found by equations (2) and (3).

If the only force acting on the body be that of gravity, and if the body start from rest in that position which makes CG horizontal, then we have

$$X = Mg\cos\theta, \quad Y = -Mg\sin\theta, \quad L = -Mgh\sin\theta;$$

$$\therefore \frac{d^2\theta}{dt^2} = -\frac{gh}{k^2 + h^2}\sin\theta,$$

integrating, we have

$$\left.\frac{d\theta}{dt}\right|^2 = C + \frac{2gh}{k^2+h^2}\cos\theta,$$

but when $\theta = \frac{\pi}{2}$, $\frac{d\theta}{dt}$ vanishes, therefore $C = 0$, substituting these values in (2) and (3), we get

$$\left.\begin{aligned} -F &= Mg\cos\theta \cdot \frac{k^2+3h^2}{k^2+h^2} \\ G &= Mg\sin\theta \cdot \frac{k^2}{k^2+h^2} \end{aligned}\right\},$$

where θ is the angle which CG makes with the vertical.

Let ψ be the angle the direction of the pressure at C makes with the line CG, the angle being measured from CG downwards to the left, then

$$\cot\psi = \left(1 + 3\frac{h^2}{k^2}\right)\cot\theta,$$

which is a convenient formula to determine the direction of the pressure*.

41. *Secondly.* Suppose either the body or the forces not to be symmetrical.

Let the fixed axis be taken as the axis of z with any origin and plane of xz. These we shall afterwards so choose as to simplify our process as much as possible. Let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the co-ordinates of the centre of gravity at the time t.

Let ω be the angular velocity of the body, f the angular acceleration, so that

$$f = \frac{d\omega}{dt}.$$

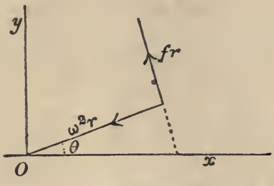
* Let mR be the resultant of F and G, and let

$$a = g\frac{k^2+3h^2}{k^2+h^2} \text{ and } b = g\frac{k^2}{k^2+h^2},$$

then
$$\frac{\cos^2\psi}{a^2} + \frac{\sin^2\psi}{b^2} = \frac{1}{R^2}.$$

Construct an ellipse with C for centre and axes equal to a and b measured along and perpendicular to CG. Then the resultant pressure varies as the diameter along which it acts. And the direction may be found thus; let the auxiliary circle cut the vertical in V, and let the perpendicular from V on CG cut the ellipse in R. Then CR is the direction of the pressure.

Now every element m of the body describes a circle about the axis, hence its accelerations along and perpendicular to



the radius vector r from the axis are $-\omega^2 r$ and fr. Let θ be the angle which r makes with the plane of xz at any time, then from the resolution of forces it is clear that

$$\frac{d^2x}{dt^2} = -\omega^2 r\cos\theta - fr\sin\theta$$

$$\left.\begin{aligned} &= -\omega^2 x - fy \\ \frac{d^2y}{dt^2} &= -\omega^2 y + fx \end{aligned}\right\}.$$

These equations may also be obtained by differentiating the equations

$$x = r\cos\theta, \quad y = r\sin\theta$$

twice, remembering that r is constant.

Conceive the body to be fixed to the axis at two points, distant a and a' from the origin, and let the reactions of the points on the body resolved parallel to the axes be respectively F, G, H; F', G', H'.

The equations of motion then give

$$\begin{aligned} \Sigma mX + F + F' &= \Sigma m\frac{d^2x}{dt^2} \\ &= \Sigma m(-\omega^2 x - fy) \\ &= -\omega^2 M\bar{x} - fM\bar{y} \quad \text{................} \quad (1), \end{aligned}$$

$$\Sigma m Y + G + G' = \Sigma m \frac{d^2y}{dt^2}$$

$$= \Sigma m(-\omega^2 y + fx)$$

$$= -\omega^2 M\bar{y} + fM\bar{x} \text{ .... } (2),$$

$$\Sigma m Z + H + H' = \Sigma m \frac{d^2z}{dt^2}$$

$$= 0 \text{ .... } (3).$$

Taking moments about the axes, we have

$$\Sigma m(yZ - zY) - Ga - G'a' = \Sigma m\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right)$$

$$= -\Sigma m\left(z\frac{d^2y}{dt^2}\right)$$

$$= \omega^2 \Sigma myz - f\Sigma mxz \text{ .... } (4),$$

by merely introducing z into the results in (2),

$$\Sigma m(zX - xZ) + Fa + F'a' = \Sigma m\left(z\frac{d^2x}{dt^2} - x\frac{d^2z}{dt^2}\right)$$

$$= -\omega^2 \Sigma mxz - f\Sigma myz \text{ ...... } (5),$$

$$\Sigma m(xY - yX) = \Sigma m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right)$$

$$= \Sigma mr^2 . \frac{d\omega}{dt}$$

$$= Mk'^2 . f \text{ .... } (6).$$

Equation (6) serves to determine f and ω, and equations (1), (2), (4), (5) then determine F, G, F', G'; H and H' are indeterminate, but their sum is given by equation (3).

Looking at these equations, we see that they would be greatly simplified in two cases.

First, if the axis of z be a principal axis at the origin,

$$\Sigma mxz = 0, \ \Sigma myz = 0,$$

and the calculation of the right-hand sides of equations (4) and (5) would only be so much superfluous labour. Hence, in attempting a problem of this kind, we should, when possible, so choose the origin that the axis of revolution is a principal axis of the body at that point.

Secondly, except the determination of f and ω by integrating equation (6), the whole process is merely an algebraic substitution of f and ω in the remaining equations. Hence our results will still be correct if we choose the plane of xz to contain the centre of gravity at the moment under consideration; this will make $\overline{y} = 0$, and thus equations (1) and (2) will be simplified.

42. Prop. *A body having one point fixed and acted on by no impressed forces is set in rotation about an axis, to determine the conditions under which it will continue to rotate about that axis.*

Let this fixed point be taken as origin, and the initial axis of rotation as the axis of z. Let this axis be supposed fixed at any other point. Now if the pressures at this second point be zero, it is evident that no force will be required to keep the axis in its place, and the body will, even if no second point be fixed, permanently rotate about this axis. The conditions therefore of a permanent axis of rotation will be found by putting $F' = 0$, $G' = 0$, $a = 0$ in the equations of the last proposition.

Since there are no impressed forces, equation (6) gives in that case $f = 0$; (4) and (5) give

$$\left.\begin{array}{l} \Sigma mxz = 0 \\ \Sigma myz = 0 \end{array}\right\},$$

(1), (2), (3) give the pressures on the fixed point.

Hence, if a body having one point fixed be set in rotation about any axis, it will not continue to rotate about that axis, unless it be a principal axis of the body at the fixed point.

If the body be entirely free, we must also have $F=0$, $G=0$, $H=0$. In this case equations (1) and (2) give $\bar{x}=0$, $\bar{y}=0$. Hence the axis of revolution must pass through the centre of gravity, and be a principal axis at every point in its length.

43. Prop. *To determine the forces which must act on a body at rest, and fixed at one point, that it may begin to rotate about any proposed straight line through that point.*

Suppose this straight line to become fixed in space at some other point, then the forces must be such that the reactions at this second point are zero.

Let the fixed point be taken as the origin, and this straight line as the axis of z, and let the plane of xz contain the centre of gravity of the body at the beginning of the motion. Then $\bar{y}=0$. Also since the body is initially at rest, $\omega=0$. Then the equations of Art. 41 become

$$\left.\begin{aligned} \Sigma m\, X + F &= 0 \\ \Sigma m\, Y + G &= 0 \\ \Sigma m\, Z + H + H' &= 0 \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\left.\begin{aligned} \Sigma m\,(yZ - zY) &= -f\,.\,\Sigma mxz \\ \Sigma m\,(zX - xZ) &= -f\,.\,\Sigma myz \\ \Sigma m\,(xY - yX) &= f\,.\,Mk'^2 \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(2).$$

The equations (1) determine only the pressures on the fixed point. The second set show that the moments of the forces about the axes must be proportional to

$$-\Sigma mxz, \quad -\Sigma myz, \quad Mk'^2.$$

The forces must therefore be equivalent to a single resultant force at the fixed point, and a single resultant couple, acting in a plane, whose equation is

$$-\Sigma mxz\,.\,X - \Sigma myz\,.\,Y + Mk'^2 Z = 0.$$

Let the momental ellipsoid at the fixed point be constructed, and let its equation be

$$AX^2 + BY^2 + CZ^2 - 2DYZ - 2EZX - 2FXY = \epsilon^4.$$

The equation to the diametral plane of the axis of z is

$$-EX - DY + CZ = 0.$$

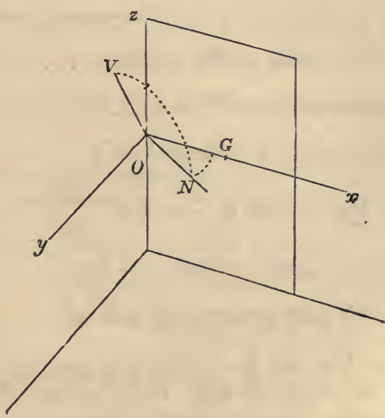
Hence the plane of the resultant couple must be the diametral plane of the axis of rotation.

The body will not in general continue to rotate about this axis.

44. Ex. 1. *A door is suspended by two hinges from a fixed axis making an angle α with the vertical. Find the motion and pressures on the hinges.*

Since the fixed axis is evidently a principal axis at the middle point, we shall take this point for origin. Also we shall take the plane of xz so that it contains the centre of gravity of the door at the moment under consideration.

The only force acting on the door is gravity, which may be supposed to act at the centre of gravity. We must first resolve this parallel to the axes. Let ϕ be the angle the plane of the door makes with a vertical plane through the



axis of suspension. If we draw a plane ZON such that its trace ON on the plane of XOY makes an angle ϕ with the axis of x, this will be the vertical plane through the axis, and if we draw OV in this plane making $ZOV = \alpha$, OV will be vertical. Hence the resolved parts of gravity are

$$X = g \sin\alpha \cos\phi, \quad Y = g \sin\alpha \sin\phi,$$
$$Z = -g \cos\alpha.$$

The six equations of motion are

$$\begin{aligned} Mg \sin\alpha \cos\phi + F + F' &= \Sigma m \frac{d^2x}{dt^2} \\ &= \Sigma m . (-\omega^2 x) \\ &= -\omega^2 M\bar{x} \ldots\ldots\ldots\ldots\ldots (1), \end{aligned}$$

$$\begin{aligned} Mg \sin\alpha \sin\phi + G + G' &= \Sigma m \frac{d^2y}{dt^2} \\ &= \Sigma m\,(fx) \\ &= fM\bar{x} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2), \end{aligned}$$

$$\begin{aligned} -Mg\cos\alpha + H + H' &= \Sigma m \frac{d^2z}{dt^2} \\ &= 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3), \end{aligned}$$

$$-Ga + G'a = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

$$Mg\cos\alpha\,\bar{x} + Fa - F'a = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

because the fixed axis is a principal axis

$$-Mg \sin\alpha \sin\phi . \bar{x} = Mk'^2 . \frac{d^2\phi}{dt^2} \ldots\ldots\ldots\ldots (6).$$

Integrating the last equation, we have

$$C + 2g \sin\alpha \cos\phi \bar{x} = k'^2 \omega^2.$$

Suppose the door to be initially placed at rest, with its plane making an angle β with the vertical plane through the axis; then when $\phi = \beta$, $\omega = 0$; hence

$$\left.\begin{aligned} k'^2\omega^2 &= 2g \sin\alpha\,(\cos\phi - \cos\beta) \\ \text{and} \qquad k'^2 f &= -g \sin\alpha \sin\phi . x \end{aligned}\right\}.$$

By substitution in the first four equations F, F', G, G', may be found.

Ex. 2. *A perfectly rough circular horizontal board is capable of revolving freely round a vertical axis through its centre. A man whose weight is equal to that of the board walks on and round it at the edge: when he has completed the circuit what will be his position in space?*

Let a be the radius of the board, Mk^2 its moment of inertia about the vertical axis. Let ω be the angular velocity of the board, ω' that of the man about the vertical axis at any time. And let F be the action between the feet of the man and the board.

The equation of motion of the board is by Art. 34

$$Mk^2 \frac{d\omega}{dt} = -Fa \qquad \text{(1)}.$$

The equation of motion of the man is by Art. 32

$$Ma \frac{d\omega'}{dt} = F \qquad \text{(2)}.$$

Eliminating F and integrating, we get

$$k^2\omega + a^2\omega' = 0,$$

the constant being nothing, because the man and board start from rest.

Let θ, θ' be the angles described by the board and man round the vertical axis.

Then $\omega = \frac{d\theta}{dt}$, $\omega' = \frac{d\theta'}{dt}$, and

$$k^2\theta + a^2\theta' = 0.$$

Hence, when $\theta' - \theta = 2\pi$, we have

$$\theta' = \frac{k^2}{k^2 + a^2} 2\pi.$$

This gives the angle in space described by the man.

If $$k^2 = \frac{a^2}{2} \text{ we have } \theta' = \frac{2}{3}\pi.$$

Let V be the mean relative velocity with which the man walks along the board. Then

$$\omega' - \omega = \frac{V}{a};$$

$$\therefore \omega = -\frac{Va}{k^2 + a^2}$$

$$= -\frac{2}{3}\frac{V}{a}.$$

This gives the mean angular velocity of the board.

45. The oscillations of a rigid body may be made use of to determine the numerical value of the accelerating force of gravity. Let L be the length of a simple equivalent pendulum of any body, and let T be the time of a complete oscillation. Then we have when the oscillations are small,

$$T = 2\pi\sqrt{\frac{L}{g}}.$$

Thus g can be determined as soon as L and T are known.

The simplest body to make use of for this purpose is a straight rod, drawn as a wire, and suspended from one extremity. It is easily proved that the centre of oscillation is at a distance from the point of suspension two-thirds of the length of the rod. Thus L is known. T may be found by comparing its oscillations with those of the pendulum of a clock.

By inverting the rod and taking the mean of the results in each position any error arising from want of uniformity in density or figure may be partially obviated. But it is very difficult to obtain a rod so uniform as to give results sufficiently accordant with each other. Captain Kater therefore proposed to use the property (Art. 37) of the convertibility of the centres of suspension and oscillation to obtain more accurate results. *Phil. Trans.* 1818.

Let a body, furnished with a moveable weight, be provided with a point of suspension C, and another point on which it may vibrate, fixed as nearly as can be estimated in the centre of oscillation O, and in a line with the point of suspension and the centre of gravity. The oscillations of the body must now be observed when suspended from C and also when suspended from O. If the vibrations in each position should not be equal in equal times, they may readily be made so by shifting the moveable weight. When this is done, the distance between the two points C and O, is the length of the simple equivalent pendulum. Thus the length L and the corresponding time T of vibration will be found uninfluenced by any irregularity of density or figure. In these experiments, after numerous trials of spheres, &c. knife edges were preferred as a means of support. At the centres of suspension and oscillation there were two triangular apertures to admit the knife edges on which the body rested while making its oscillations.

46. Having thus the means of measuring the length L with accuracy, it remains to determine the time T. This is effected by comparing the vibrations of the body with those of a clock. The time of a single vibration or of any small arbitrary number of vibrations cannot be observed directly, because this would require the fraction of a second of time, as shown by the clock, to be estimated either by the eye or ear. The vibrations of the body may be counted, and here there is no fraction to be estimated, but these vibrations will not probably fit in with the oscillations of the clock pendulum, and the differences must be estimated. This defect is overcome by "the method of coincidences." Supposing the time of vibration of the clock to be a little less than that of the body, the pendulum of the clock will gain on the body, and at length at a certain vibration the two will for an instant coincide. The two pendulums will now be seen to separate and after a time will again approach each other, when the same phenomenon will take place. If the two pendulums continued to vibrate with perfect uniformity, the number of oscillations of the pendulum of the clock in this interval will be an integer, and the number of oscillations of the body in the same interval will be less by one complete oscillation

than that of the pendulum of the clock. Hence by a simple proportion the time of a complete oscillation may be found.

The coincidences were determined in the following manner. Certain marks made on the two pendulums were observed by a telescope at the lowest point of their arcs of vibration. The field of view was limited by a diaphragm to a narrow aperture across which the marks were seen to pass. At each succeeding vibration the clock pendulum follows the other more closely, and at last the clock-mark completely covers the other during their passage across the field of view of the telescope. After a few vibrations it appears again preceding the other. The time of disappearance was generally considered as the time of coincidence of the vibrations, though in strictness the mean of the times of disappearance and reappearance ought to have been taken, but the error thus produced is evidently very small. *Encyc. Met.* Figure of the Earth. In the experiments made in Harton coal-pit in 1854, the Astronomer Royal used Kater's method of observing the pendulum, (*Phil. Trans.* 1856.)

The value of T thus found will require several corrections. These are called "Reductions." If the centre of oscillation do not describe a cycloid, allowance must be made for the alteration of time depending on the arc described. This is called "the reduction to infinitely small arcs." If the point of support be not absolutely fixed, another correction is required (*Phil. Trans.* 1831). The effect of the buoyancy and the resistance of the air must also be allowed for. This is the "reduction to a vacuum." The length L must also be corrected for changes of temperature.

The time of an oscillation thus corrected enables us to find the value of gravity at the place of observation. A correction is now required to reduce this result to what it would have been at the level of the sea. The attraction of the intervening land must be allowed for by Dr Young's rule (*Phil. Trans.* 1819). We thus obtain the force of gravity at the level of the sea, supposing all the land above this level were cut off and the sea constrained to keep its present level. As the sea would tend in such a case to change its level, further corrections are still necessary if we wish to reduce the result to the surface of that spheroid which most nearly represents the earth. See *Camb. Phil. Trans.* Vol. x.

47. There is another use to which the experimental determination of the length of a simple equivalent pendulum may be applied. "It has been adopted as a standard of length on account of being invariable and capable at any time of recovery. An Act of Parliament, 5 Geo. IV. defines the yard to contain 36 such parts, of which parts there are 39·1393 in the length of a pendulum vibrating seconds of mean time in the latitude of London in vacuo at the level of the sea at temperature 62° F. The Commissioners, however, appointed to consider the mode of restoring the standards of weight and measure which were lost by fire in 1834, report that several elements of reduction of pendulum experiments are yet doubtful or erroneous, so that the results of a convertible pendulum are not so trustworthy as to serve for supplying a standard of length; and they recommend a material standard, the distance namely between two marks on a certain bar of metal under given circumstances, in preference to any standard derived from measuring phenomena in nature (*Report*, 1841)." Griffin's *Dynamics of a Rigid Body*, page 24.

EXAMPLES.

1. Find the time of the small oscillations of a cube (1) when one side is fixed, (2) when the diagonal of one of its faces is fixed; the axis in both cases being horizontal.

Result. The length of the simple equivalent pendulum is in the first case $\frac{4\sqrt{2}}{3}a$, in the second $\frac{5}{3}a$, where $2a$ is the side of the cube.

2. Find the eccentricity of an elliptic lamina such that when it swings about one latus rectum, the other latus rectum may pass through the centre of oscillation.

Result. The eccentricity $= \frac{1}{2}$.

3. A circular arc oscillates about an axis through its middle point perpendicular to the plane of the arc. Prove that the length of the simple equivalent pendulum is independent of the length of the arc and is equal to twice the radius of the arc.

4. A heavy uniform quadrant is attached to a horizontal axis at the extremities of one of its bounding radii, and revolves about it; determine the action on the points of support, the quadrant being originally horizontal.

5. Find what axis in the area of an ellipse must be fixed in order that the time of a small oscillation may be a minimum.

6. The density of a rod varies as the distance from one end. Find the axis perpendicular to it about which the time of oscillation is a minimum.

Result. The axis passes through either of two points whose distance from the centre of gravity is $\frac{\sqrt{2}}{6}a$, where a is the length of the rod.

7. A rod is fixed at one extremity to an axis about which it can freely oscillate. The angle between the rod and the axis is α, find the angle β which the axis must make with the vertical in order that the rod may oscillate in n seconds of time.

8. Find the time of oscillation of a rectangle about a horizontal axis passing through the middle point of the upper side, perpendicular to the line through that point bisecting the rectangle, and making an angle α with the upper side.

9. A pyramid whose base is an equilateral triangle and whose faces are given isosceles triangles being made to turn about a fixed edge of its base, is left to itself immediately after passing the position of unstable equilibrium; find the angular velocity in any position.

10. The centre of oscillation of a pendulum is retarded by a constant force $= ng$: prove that in small oscillations the decrement of the angle of ascent is nearly equal to $2n$.

11. In the case of motion about a horizontal axis under the action of gravity, shew that the forces are reducible to a single force, if the axis be a principal axis at the point where the perpendicular upon it from the centre of gravity meets it; and not otherwise. If the axis be a principal axis, but at another point in it, and if the centre of gravity start from the horizontal plane passing through the axis; determine the pressures.

12. A uniform rod is revolving about a horizontal axis which passes through it at a distance b from the further end. Shew that the tendency to break is greatest at a point of the rod distant from that end double the difference between b and the length of the equivalent isochronous simple pendulum for all positions of the rod.

13. A rod is inclined at an angle of 30^0 to an axis about which it revolves with uniform angular velocity. Supposing gravity to be neglected, compare the tendencies to break at different points of the rod.

14. A uniform beam, moveable about a hinge at one extremity, is supported at the other by an elastic string fastened to a point at a distance c vertically above the hinge, so that the string and beam are, in the position of equilibrium, at right angles. Shew that if in that position the stretched length of the string is twice the natural length, the simple equivalent pendulum is of length $\frac{2c}{3}$.

15. A uniform stick hangs freely by one end, the other end being close to the ground. An angular velocity in a vertical plane is then communicated to the stick, and when it has risen through an angle of 90^0, the end by which it was hanging is loosed. What must be the initial angular velocity

so that on falling to the ground it may pitch in an upright position?

16. A uniform circular plate radius a is capable of revolving about a smooth horizontal axis through its centre, a rough string equal in mass to the plate and in length to its circumference hangs over its rim in equilibrium; shew that the velocity of the string when it begins to leave the plate is $\sqrt{\frac{g\pi a}{6}}$.

CHAPTER IV.

MOTION IN TWO DIMENSIONS.

SECT. I. *Fixed Axes.*

48. THERE are two principles which will in general conduct us to the solution of every dynamical problem. These have been already demonstrated and may be briefly enunciated thus.

(1) The motion of the centre of gravity of a rigid body is the same as if all the mass were collected at that point and was acted on by all the forces applied at that point parallel to their original directions.

(2) The motion of rotation round the centre of gravity is the same as if that point were fixed.

The first of these principles enables us to write down the equations of the motion of the centre of gravity. The question is one of Dynamics of a Particle. There are three sets of equations which we may use.

First. The equations in rectilinear co-ordinates,

$$\frac{d^2\bar{x}}{dt^2} = X, \quad \frac{d^2\bar{y}}{dt^2} = Y.$$

Secondly. The equations in polar co-ordinates,

$$\frac{d^2\bar{r}}{dt^2} - \bar{r}\left.\overline{\frac{d\bar{\theta}}{dt}}\right|^2 = P, \quad \frac{1}{\bar{r}}\frac{d}{dt}\left(\bar{r}^2\frac{d\bar{\theta}}{dt}\right) = Q.$$

Thirdly. The equations expressing the accelerations along the tangent and normal,

$$\frac{d^2\bar{s}}{dt^2} = S, \quad \frac{\bar{v}^2}{\bar{\rho}} = N,$$

where S and N are the resolved parts of impressed forces along the tangent and normal in the directions in which $\bar{s}$ and $\bar{\rho}$ are respectively measured. The quantities $\bar{x}$, $\bar{y}$, $\bar{r}$, $\bar{\theta}$, are the co-ordinates of the centre of gravity, $\bar{v}$ its velocity, and $\bar{\rho}$ the radius of curvature of its path.

The second principle will enable us to regard the centre of gravity as a fixed point. The motion is then reduced to that about a fixed axis. The equation to such a motion is

$$\frac{d^2\theta}{dt^2} = \frac{\text{moment of forces}}{\text{moment of inertia}},$$

where θ is the angle made by any fixed straight line in the body with any fixed straight line in space.

These considerations will furnish us with three equations of motion for every rigid body in the system under consideration. These may be called the Dynamical equations. It is obvious that as a body in two dimensions can admit of only three independent motions, no more than three equations to determine these can be obtained. These three independent motions are the two velocities of translation parallel to the axes and the velocity of rotation.

Besides these there will be certain geometrical equations expressing the given connections of the system. As every such forced connection is necessarily accompanied by a reaction and every reaction by some forced connection, the number of geometrical equations will also be the same as the number of unknown reactions in the problem. If however any subsidiary quantity has been introduced there must be a new geometrical equation for every such quantity.

49. Having obtained a sufficient number of equations of motion we must proceed to their solution. Two general methods have been proposed.

First Method. Differentiate the geometrical equations twice with respect to t, and substitute for $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2\theta}{dt^2}$, from the dynamical equations. We shall then have a sufficient number of equations to determine the reactions. This method will be of great advantage whenever the geometrical equations are of the form

$$Ax + By + C\theta = D \text{..................(1)},$$

where A, B, C, D are constants. Suppose also that the dynamical equations are such that when written in the form

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \&c. \\ \frac{d^2y}{dt^2} &= \&c. \end{aligned}\right\} \text{..........................(2)}$$

they contain only the reactions and constants on the right-hand side without any x, y, or θ. Then, when we substitute in the equation

$$A\frac{d^2x}{dt^2} + B\frac{d^2y}{dt^2} + C\frac{d^2\theta}{dt^2} = 0,$$

obtained by differentiating (1), we have an equation containing only the reactions and constants. This being true for all the geometrical relations, it is evident that all the reactions will be constant throughout the motion and their values may be found. Hence when these values are substituted in the dynamical equations (2), their right-hand members will all be constants and the values of x, y, and θ may be found by an easy integration.

If however the geometrical equations are not of the form (1), this method of solution will usually fail. For suppose any geometrical equation took the form

$$x^2 + y^2 = c^2,$$

containing *squares* instead of *first* powers, then its second differential equation will be

$$x\frac{d^2x}{dt^2}+y\frac{d^2y}{dt^2}+\overline{\frac{dx}{dt}}\Big|^2+\overline{\frac{dy}{dt}}\Big|^2=0;$$

and though we can substitute for $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, we cannot in general eliminate the terms $\overline{\frac{dx}{dt}}\Big|^2$ and $\overline{\frac{dy}{dt}}\Big|^2$.

50. *Second Method.* Suppose the system to consist of only one body of mass m, and let the equations of motion be written in the form

$$\left.\begin{aligned} m\frac{d^2x}{dt^2}&=AR+\dots\\ m\frac{d^2y}{dt^2}&=BR+\dots\\ mk^2\frac{d^2\theta}{dt^2}&=CR+\dots\end{aligned}\right\}\dots\dots\dots\dots\dots\dots(2),$$

where A, B, C are the coefficients of some unknown reaction R which may enter into all the equations. Multiplying these equations respectively by $2\frac{dx}{dt}$, $2\frac{dy}{dt}$ and $2\frac{d\theta}{dt}$, and adding, we get

$$2m\frac{dx}{dt}\frac{d^2x}{dt^2}+2m\frac{dy}{dt}\frac{d^2y}{dt^2}+2mk^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2}$$

$$=2\left(A\frac{dx}{dt}+B\frac{dy}{dt}+C\frac{d\theta}{dt}\right)R+\dots$$

then it will be proved in a subsequent chapter that, by virtue of the geometrical equations, the coefficient of R will vanish. And in the same way all the other unknown reactions will disappear from the equation.

Integrating this equation we get

$$m\left(\frac{dx}{dt}\right)^2 + m\left(\frac{dy}{dt}\right)^2 + mk^2\left(\frac{d\theta}{dt}\right)^2 = \text{known function of } x,\ y,$$

and θ.

If there be two geometrical equations we shall be able to express x and y in terms of θ, and substituting we shall have

$$\frac{d\theta}{dt} = \text{known function of } \theta.$$

This when solved will enable us to determine x, y, and θ as functions of t.

If there be only one geometrical equation there will be only one unknown reaction in the original equations (2). This must be eliminated from the equations by some process different from that described above, and adapted to the particular case in question. It is obvious there cannot be more than two independent geometrical relations, for then no motion would in general be possible.

If there be several bodies in the proposed Dynamical System, the same process will apply. Each set of equations must be multiplied by the factors above described and all the sets must be added together.

This method, with a few exceptions, will give a first integral of the original equations free from any unknown reactions. If the whole system of bodies be so connected by its geometrical relations that only one independent motion of the whole system is possible this one equation will be sufficient to give that motion.

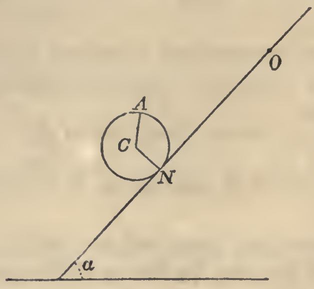
This is called the method of *vis viva*, and the cases of exception will be considered under that head.

51. Ex. 1. *A sphere whose centre of gravity is in its centre rolls down a perfectly rough inclined plane. Find the motion.*

Let α be the inclination of the plane to the horizon, a the radius of the sphere, mk^2 its moment of inertia about a horizontal diameter.

Let O be that point of the inclined plane at which the sphere originally started, and N the point of contact at time t.

Then it obviously is best to choose O for origin and ON for the axis of x. The forces which act on the sphere are first,



the reaction R perpendicular to ON, secondly, F the friction acting at N along NO, and mg acting vertically at C the centre.

Then the first principle gives the equations

$$\frac{d^2x}{dt^2} = g \sin \alpha - \frac{F}{m} \quad \text{.......................(1)},$$

$$\frac{d^2y}{dt^2} = -g \cos \alpha + \frac{R}{m} \quad \text{...................(2)}.$$

The second principle gives

$$\frac{d^2\theta}{dt^2} = \frac{Fa}{mk^2} \quad \text{...............................(3)},$$

where θ may be taken to be the angle which the radius whose extremity originally coincided with O, makes with the normal to the inclined plane. Then θ is the angle turned through by the sphere.

Since there are two unknown reactions, F and R, we require two geometrical equations. Since there is no slipping we have

$$x = a\theta \quad \text{...............................(4)}.$$

Also since there is no jumping

$$y = a \text{ ....(5).}$$

Both these equations are of the form described in the first method. Hence differentiating twice

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= a\frac{d^2\theta}{dt^2} \\ \frac{d^2y}{dt^2} &= 0 \end{aligned}\right\}.$$

Substituting from (1), (2), (3),

$$\left.\begin{aligned} g \sin\alpha - \frac{F}{m} &= \frac{Fa^2}{mk^2} \\ g \cos\alpha - \frac{R}{m} &= 0 \end{aligned}\right\}.$$

Hence

$$\left.\begin{aligned} F &= m \cdot \frac{k^2}{k^2 + a^2} \cdot g \sin\alpha \\ R &= mg \cos\alpha \end{aligned}\right\} \text{.............(6).}$$

If the sphere be homogeneous $k^2 = \frac{2}{5} a^2$, and we have

$$F = \frac{2}{7} mg \sin\alpha.$$

Hence

$$\frac{d^2x}{dt^2} = \frac{5}{7} g \sin\alpha.$$

If the sphere had been a particle sliding down a *smooth* plane the equation of motion would have been

$$\frac{d^2x}{dt^2} = g \sin\alpha.$$

So that the acceleration of the rolling sphere is just $\frac{5}{7}$ of that of the particle. Supposing the sphere to start initially from rest, we have clearly

$$x = \frac{1}{2} \cdot \frac{5}{7} g \sin \alpha \,.\, t^2,$$

and the whole motion is determined.

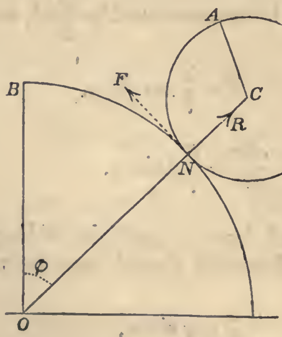
It is usual to delay the substitution of the value of k^2 in the equations until the end of the investigation, for this value is often very complicated. But there is another advantage. It serves as a verification of the signs in our original equations, for if equation (6) had been

$$F = m \frac{k^2}{k^2 - a^2} g \sin \alpha,$$

we should have expected some error, for it seems clear that the friction could not be made infinite by any alteration of the internal structure of the sphere.

52. Ex. 2. *A sphere rolls down another perfectly rough fixed sphere. Find the motion.*

Let a and b be the radii of the moving and fixed sphere, respectively, C and O the two centres.



Let OB be a vertical radius of the fixed sphere, and $\phi = \angle BOC$. Let F and R be the friction and normal reaction at N. Then taking the tangential and normal resolutions, the equations for the motion of C are

$$(a+b)\frac{d^2\phi}{dt^2} = g\sin\phi - \frac{F}{m} \quad\text{.................(1)},$$

$$(a+b)\left.\frac{d\phi}{dt}\right|^2 = g\cos\phi - \frac{R}{m} \quad\text{.................(2)}.$$

Let A be that point of the moving sphere which originally coincided with B. Then if θ be the angle which any fixed line CA in the body makes with any fixed line in space as the vertical, we have

$$\frac{d^2\theta}{dt^2} = \frac{Fa}{mk^2} \quad\text{..............................(3)}.$$

It should be observed that we cannot take θ as the angle ACO because, though CA is fixed in the body, CO is not fixed in space.

The geometrical equation is clearly

$$a(\theta - \phi) = b\phi \quad\text{.......................(4)}.$$

No other is wanted, since in forming equations (1) and (2) the constancy of the distance CO has been already supposed.

The form of equation (4) shews that we can apply the first method. We thus obtain

$$F = \frac{k^2}{k^2+a^2} mg \sin\phi,$$

and we are finally led to the equation

$$(a+b)\frac{d^2\phi}{dt^2} = \frac{5}{7} g\sin\phi.$$

By multiplying by $2\frac{d\phi}{dt}$ and integrating we get after determining the constant

$$\left.\frac{d\phi}{dt}\right]^2 = \frac{10}{7}\frac{g}{a+b}(1-\cos\phi),$$

the rolling body being supposed to start from rest at a point indefinitely near B.

To find where the body leaves the sphere we must put $R=0$. This gives by (2)

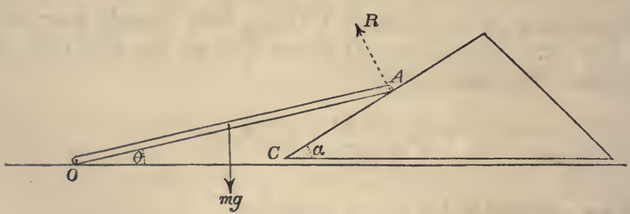
$$(a+b)\left.\frac{d\phi}{dt}\right]^2 = g\cos\phi;$$

$$\therefore\ \frac{10}{7}g(1-\cos\phi) = g\cos\phi;$$

$$\therefore\ \cos\phi = \frac{10}{17}.$$

53. Ex. 3. *A rod OA can turn about a hinge at O, while the end A rests on a smooth wedge which can slide along a smooth horizontal plane through O. Find the motion.*

Let $\alpha=$ the inclination of the wedge, $M=$ its mass and $x=OC$.



Let $l=$ the length of the beam, $m=$its mass and $\theta=AOC$. Let $R=$ the reaction at A. Then we have

the dynamical equations,

$$\frac{d^2x}{dt^2} = \frac{R\sin\alpha}{M} \quad\dots\dots\dots\dots(1),$$

$$\frac{d^2\theta}{dt^2} = \frac{Rl\,.\cos(\alpha-\theta) - mg\,\frac{l}{2}\cos\theta}{mk^2} \;\ldots\ldots\ldots(2),$$

and the geometrical equation,

$$x = \frac{l}{\sin\alpha}\,.\sin(\alpha-\theta) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

It is obvious we must apply the second method of solution. Hence

$$2M\frac{dx}{dt}\frac{d^2x}{dt^2} + 2mk^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} = -mgl\cos\theta\frac{d\theta}{dt}$$

$$+ 2R\left\{\sin\alpha\frac{dx}{dt} + l\cos(\alpha-\theta)\frac{d\theta}{dt}\right\}.$$

The coefficient of R is seen to vanish by differentiating equation (3). Integrating we have

$$M\left(\frac{dx}{dt}\right)^2 + mk^2\left(\frac{d\theta}{dt}\right)^2 = C - mgl\sin\theta.$$

Substituting from (3) we have

$$\left\{M\frac{l^2}{\sin^2\alpha}\cos^2(\alpha-\theta) + mk^2\right\}\left(\frac{d\theta}{dt}\right)^2 = C - mgl\sin\theta.$$

If the beam start from rest when $\theta = \beta$, then $C = mgl\sin\beta$.

This equation cannot be integrated any further. We cannot therefore find θ in terms of t. But the angular velocity of the beam, and therefore the velocity of the wedge, is given by the above equation.

54. The nature of the action of one part OP of the rod on the remaining part PA may be found as follows.

When a rod is under the action of forces in equilibrium, we know from Statics that this action is equivalent to the resultants of all the forces which act on *one side* of P.

These will generally be equivalent to a force and a couple. The latter is the most important part of the action, and is called the "Tendency to Break."

Let du be any element of the rod distant u from P, and on the side of P nearer the end A of the rod, and let $PA = z$. The effective moving forces on du are

$$m\frac{du}{l}\cdot u\frac{d^2\theta}{dt^2} \text{ and } m\frac{du}{l}\cdot u\left.\overline{\frac{d\theta}{dt}}\right|^2$$

respectively perpendicular to and along the rod. The impressed force is $m\dfrac{du}{l}\cdot g$. The effective forces being reversed, the tendency to break at P is equal to the moment about P, of all the forces which act on the part PA of the rod. If this be called L, we have

$$L = \int_0^z \left\{m\frac{du}{l}\cdot\frac{d^2\theta}{dt^2}u^2 + m\frac{du}{l}gu\cos\theta\right\} - Rz$$

$$= m\frac{z^3}{3l}\cdot\frac{d^2\theta}{dt^2} + m\frac{gz^2}{2l}\cos\theta - Rz,$$

and the values of $\dfrac{d^2\theta}{dt^2}$ and R must be substituted.

55. When any of the bodies in the system are not perfectly rough, we know that the friction will be different according as the body only rolls, or partly rolls and partly slides. The difficulty in such a case is to determine which of these two possible motions is the actual one. The usual method is to take the friction to be F, assuming the body to roll. Then solving the equations on this supposition, we can determine the ratio $\dfrac{F}{R}$, where R is the normal reaction. If this ratio be less than μ, the supposition was correct. If not there will be sliding, and we should take the friction $= \mu R$.

Ex. 4. *A sphere is placed on a fixed rough inclined plane, the coefficient of friction being μ, determine whether the sphere will slide or roll.*

Taking the same figure and notation as in Example 1, we have

$$\frac{F}{R}=\frac{2}{7}\tan\alpha,$$

if then $\mu>\frac{2}{7}\tan\alpha$, the solution there given is correct. If μ be less than this value, the equations will be

$$\frac{d^2x}{dt^2}=g\sin\alpha-\frac{\mu R}{m},$$

$$0=g\cos\alpha-\frac{R}{m},$$

$$\frac{d^2\theta}{dt^2}=\frac{\mu Ra}{mk^2};$$

whence we have

$$\left.\begin{aligned}\frac{d^2x}{dt^2}&=g(\sin\alpha-\mu\cos\alpha)\\ \frac{d^2\theta}{dt^2}&=\frac{\mu ga}{k^2}\cdot\cos\alpha\end{aligned}\right\},$$

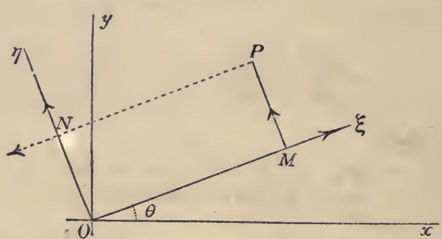
and the motion is evidently one of constant acceleration.

SECT. II. *Moving Axes.*

56. There are many problems in Rigid Dynamics which can be easily solved by a proper use of moving axes. We shall now proceed to determine the equations of motion of a body with reference to axes moving in any manner whatever. Two sets of equations are clearly necessary, first the equations expressing the motion of translation, and secondly those expressing the motion of rotation.

57. PROP. I. *To determine the equations of motion of a point with reference to two rectangular co-ordinate axes ξ, η, moving according to a given law, the origin being supposed fixed.*

Let Ox, Oy, be any two fixed axes; $O\xi$, $O\eta$, moving axes,



and let the angle $xO\xi = \theta$, then since $OM = \xi$, $PM = \eta$, we have

$$x = \xi \cos\theta - \eta \sin\theta;$$

$$\therefore \frac{dx}{dt} = \frac{d\xi}{dt}\cos\theta - \frac{d\eta}{dt}\sin\theta - (\xi\sin\theta + \eta\cos\theta)\frac{d\theta}{dt};$$

$$\therefore \frac{d^2x}{dt^2} = \frac{d^2\xi}{dt^2}\cos\theta - \frac{d^2\eta}{dt^2}\sin\theta - 2\left(\frac{d\xi}{dt}\sin\theta + \frac{d\eta}{dt}\cos\theta\right)\frac{d\theta}{dt}$$

$$-(\xi\cos\theta - \eta\sin\theta)\left(\frac{d\theta}{dt}\right)^2 - (\xi\sin\theta + \eta\cos\theta)\frac{d^2\theta}{dt^2}.$$

This equation gives the acceleration of P parallel to any fixed straight line Ox. If then we so choose the axis of x that the straight line $O\xi$ is just passing through it at the moment under consideration, this equation will also give the acceleration along $O\xi$ at that instant.

But at that instant $\theta = 0$; hence

$$\frac{d^2x}{dt^2} = \frac{d^2\xi}{dt^2} - \xi\left(\frac{d\theta}{dt}\right)^2 - \frac{1}{\eta}\frac{d}{dt}\left(\eta^2\frac{d\theta}{dt}\right) \text{.........} (1),$$

is the acceleration along $O\xi$. And as the same reasoning will apply at every other instant this must be always the acceleration along $O\xi$.

Similarly by putting $\theta = -\frac{\pi}{2}$, we can show that the acceleration along $O\eta$ is

$$\frac{d^2x}{dt^2} = \frac{d^2\eta}{dt^2} - \eta\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{\xi}\cdot\frac{d}{dt}\left(\xi^2\frac{d\theta}{dt}\right) \text{.........} (2).$$

And the equations of motion can be formed by equating these two expressions to the resolved parts of the forces in the two directions $O\xi$, $O\eta$.

58. There is another demonstration which may be given of these two equations which is so simple that it will at once enable us to remember them.

It is evident that the motion of P is made up of the motions of the two points M and N by simple additions. But the accelerations of M are

$$\left.\begin{array}{l} \dfrac{d^2\xi}{dt^2} - \xi\left(\dfrac{d\theta}{dt}\right)^2 \text{ along } OM \\[2ex] \dfrac{1}{\xi}\dfrac{d}{dt}\left(\xi^2\dfrac{d\theta}{dt}\right) \text{ perpendicular to } OM \end{array}\right\},$$

and the accelerations of N are

$$\left.\begin{array}{l} \dfrac{d^2\eta}{dt^2} - \eta\left(\dfrac{d\theta}{dt}\right)^2 \text{ along } ON \\[2ex] \dfrac{1}{\eta}\dfrac{d}{dt}\left(\eta^2\dfrac{d\theta}{dt}\right) \text{ perpendicular to } ON \end{array}\right\}.$$

Hence adding these together with their proper signs as shown by the arrows in the figure, we have

$$\frac{d^2\xi}{dt} - \xi\left(\frac{d\theta}{dt}\right)^2 - \frac{1}{\eta}\frac{d}{dt}\left(\eta^2\frac{d\theta}{dt}\right) = \text{acceleration along } O\xi.$$

$$\frac{d^2\eta}{dt^2} - \eta\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{\xi}\frac{d}{dt}\left(\xi^2\frac{d\theta}{dt}\right) = \text{acceleration along } O\eta^*.$$

59. If the moving axes revolve with uniform angular velocity ω, the above equations take the simple form

$$\frac{d^2\xi}{dt^2} - \omega^2\xi - 2\omega\frac{d\eta}{dt} = X,$$

$$\frac{d^2\eta}{dt^2} - \omega^2\eta + 2\omega\frac{d\xi}{dt} = Y,$$

where X and Y are the resolved parts of the accelerating forces along the moving axes.

60. It will frequently be found convenient to use Greek letters to express the co-ordinates of a particle when referred to *moving* axes, and English letters when referred to *fixed* axes.

61. Ex. *A particle slides along a smooth curve which turns with uniform angular velocity ω about a fixed point O. To find the motion of the particle.*

* These equations were first published in Vol. X. of the *Cambridge Phil. Trans.* They have been independently obtained by several persons, and were again published in the *Quarterly Journal of Mathematics*, 1858.

Take $O\xi$, $O\eta$ axes moving with the curve, and let R be the reaction at any point. Then we have the equations

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2} - \omega^2\xi - \frac{1}{\eta}\frac{d}{dt}(\omega\eta^2) &= -\frac{R}{m}\frac{d\eta}{ds}\\ \frac{d^2\eta}{dt^2} - \omega^2\eta + \frac{1}{\xi}\frac{d}{dt}(\omega\xi^2) &= \frac{R}{m}\frac{d\xi}{ds}\end{aligned}\right\}.$$

Let v be the velocity relative to the curve, then remembering that ω is constant, these equations reduce to

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2} &= \omega^2\xi - \left(\frac{R}{m} - 2\,\omega v\right)\frac{d\eta}{ds}\\ \frac{d^2\eta}{dt^2} &= \omega^2\eta + \left(\frac{R}{m} - 2\omega v\right)\frac{d\xi}{ds}\end{aligned}\right\}.$$

These are the equations of motion of a particle moving along a *fixed* curve and acted on by a repulsive force $\omega^2 r$ tending from O, with $\frac{R}{m} - 2\omega v$ written for R.

To find the motion we may therefore treat the curve as fixed. Hence resolving along the tangent

$$v\frac{dv}{ds} = \omega^2 r\frac{dr}{ds},$$

$$\therefore\ v^2 = c^2 + \omega^2 r^2,$$

where r is the distance of the particle from O.

Also, resolving along the normal

$$\frac{v^2}{\rho} = -\,\omega^2 r\sin\phi + \left(\frac{R}{m} - 2\omega v\right),$$

where ϕ is the angle r makes with the tangent. If p be the perpendicular drawn from O on the tangent, we have

$$\frac{R}{m} = \frac{v^2}{\rho} + \omega^2 p + 2\omega v.$$

If ω be not constant, the equation for v cannot be integrated, but the expression for R is

$$\frac{R}{m} = \frac{v^2}{\rho} + \omega^2 p + 2\omega v + \frac{d\omega}{dt}\sqrt{(r^2 - p^2)}.$$

62. PROP. II. *To determine the equations of motion of a point with reference to polar co-ordinates, the origin being supposed to move according to any given law.*

Let Ox, Oy be any two fixed axes, C the moving origin and P the point. Let $CP = r$, and let CP make an angle θ with any fixed straight line Ox.

Since the relative motion only, of P with respect to C is required, we may reduce C to rest by applying to both C and P accelerations equal and opposite to the accelerations of C. The particle P will then be acted on by the accelerating forces given by the question and by the reversed accelerations of the point C. Hence we shall have

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \left\{\begin{array}{l}\text{impressed accelerating force on } P \text{ along } CP \\ \text{plus reversed acceleration of } C \text{ resolved along} \\ CP,\end{array}\right.$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \left\{\begin{array}{l}\text{impressed accelerating force on } P \text{ perpendi-} \\ \text{cular to } CP \text{ plus reversed acceleration of } C \\ \text{perpendicular to } CP.\end{array}\right.$$

63. PROP. III. *To determine the equations of motion of a point in terms of its velocities parallel to two co-ordinate axes which move according to any given law.*

Let Ox, Oy be any two fixed axes; $O\xi$, $O\eta$ the moving directions, and let as before the angle $xO\xi = \theta$. Let u, v, be the velocities of the point parallel to the axes $O\xi$, $O\eta$. Then

$$\frac{dx}{dt} = u\cos\theta - v\sin\theta;$$

$$\therefore \quad \frac{d^2x}{dt^2} = \left(\frac{du}{dt} - v\frac{d\theta}{dt}\right)\cos\theta - \left(u\frac{d\theta}{dt} + \frac{dv}{dt}\right)\sin\theta.$$

Then by the same reasoning as in Art. 57, putting first $\theta = 0$ and then $\theta = -\frac{\pi}{2}$, we get the accelerations parallel to $O\xi$, $O\eta$. Hence if X and Y be the impressed accelerating forces in these directions, the equations of motion are

$$\left.\begin{aligned}\frac{du}{dt} - v\frac{d\theta}{dt} = X,\\ \frac{dv}{dt} + u\frac{d\theta}{dt} = Y.\end{aligned}\right\}.$$

64. PROP. IV. *To determine the equations of rotation of a body about a point which moves according to any given law.*

Let Ox, Oy be any fixed axes, and let the co-ordinates of the moving point C be p, q.

Let
$$\left.\begin{aligned}x = p + \xi\\ y = q + \eta\end{aligned}\right\}.$$

Then the equation of motion is

$$\Sigma m \left\{(p+\xi)\frac{d^2(q+\eta)}{dt^2} - (q+\eta)\frac{d^2(p+\xi)}{dt^2}\right\}$$
$$= \text{moment of the forces about } O.$$

Since the position of the fixed point O is quite arbitrary, we may take it so that the moving origin is passing through it at the instant under consideration. At this instant we have $p = 0$, $q = 0$. Hence

$$\frac{d^2q}{dt^2}\Sigma m\xi - \frac{d^2p}{dt^2}\Sigma m\eta + \Sigma m\left(\xi\frac{d^2\eta}{dt^2} - \eta\frac{d^2\xi}{dt^2}\right)$$
$$= L, \text{ the moment of the forces about } C.$$

Let $\bar{\xi}$, $\bar{\eta}$ be the co-ordinates of the centre of gravity referred to axes through C parallel to the axes Ox, Oy; then the above equation becomes

$$\Sigma m\left(\xi\frac{d^2\eta}{dt^2} - \eta\frac{d^2\xi}{dt^2}\right) = L + \left(\frac{d^2p}{dt^2}.\bar{\eta} - \frac{d^2q}{dt^2}\bar{\xi}\right).\Sigma m \ldots(A).$$

65. There is another demonstration which can be given of this proposition which will enable us to remember the result.

Since only the relative motion of the body and the point C is required, we may reduce C to rest by applying to every element of the body an acceleration equal and opposite to that of C. Let now r be the distance of any particle of the body from C, $\frac{d\theta}{dt}$ its angular velocity round C. Then the accelerations of the particle will be

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \text{ and } \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

along and perpendicular to r. Then reversing these and taking moments about C, we have by D'Alembert's Principle,

$$\Sigma m \frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \begin{cases}\text{moment of the impressed forces plus the}\\ \text{moment of the added forces.}\end{cases}$$

If every particle m of a body be acted on by the same force f always acting parallel to a fixed direction, it is evident that the sum of the moments of all these forces about any point is equal to the moment of $f\Sigma m$ supposed to act at the centre of gravity. This is true whatever f may be and is still true if for f we read the reversed acceleration of C. Hence we have

$$\Sigma m \frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \begin{cases}\text{moment round } C \text{ of the impressed forces}\\ \text{plus the moment round } C \text{ of the reversed}\\ \text{effective force of } C \text{ supposed to act at the}\\ \text{centre of gravity} \ldots\ldots\ldots\ldots (B).\end{cases}$$

By a well-known transformation we have

$$\xi\frac{d\eta}{dt} - \eta\frac{d\xi}{dt} = r^2\frac{d\theta}{dt},$$

and therefore $$\xi\frac{d^2\eta}{dt^2} - \eta\frac{d^2\xi}{dt^2} = \frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right);$$

hence the two equations (A) and (B) are exactly the same.

66. If the point C be fixed in the body and move with it in space, $\frac{d\theta}{dt}$ will be the same for every element of the body, and r will be constant throughout the motion; hence

$$\Sigma m \frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) = \Sigma m r^2 \frac{d^2\theta}{dt^2}$$

$$= Mk^2 \frac{d\omega}{dt},$$

where ω is the angular velocity of the body, and θ is the angle made by a fixed line in the body with a fixed line in space.

67. From the general equation of moments about a moving point we learn that we may use the equation

$$\frac{d\omega}{dt} = \frac{\text{moment of forces about } C}{\text{moment of inertia about } C}$$

in the following cases.

First. If the point C be fixed both in the body and in space; for then the acceleration of C is nothing.

Secondly. If the point C being fixed in the body move in space with uniform velocity; for the same reason as before.

Thirdly. If the point C be the centre of gravity; for in that case $\bar{\xi} = 0$, $\bar{\eta} = 0$.

Fourthly. If the point C be the instantaneous centre of rotation*, and the motion be a small oscillation. At the time t the body is turning about C, and the velocity of C is therefore zero. At the time $t + dt$, the body is turning about some point C' very near to C. Let $CC' = d\sigma$, then the

* If a body be in motion in one plane it is known that the actual displacement of every particle in the time dt is the same as if the body had been turned through some angle ωdt about some fixed point C. This may be proved in the same way as the corresponding proposition in Three Dimensions is proved in the next Section. See Prop. 1. The point C is called the instantaneous centre of rotation, and ω is called the instantaneous angular velocity. See also Salmon's *Higher Plane Curves*, 1852, Arts. 246 and 264.

velocity of C is $\omega d\sigma$. Hence in the time dt the velocity of C has increased from zero to $\omega d\sigma$, therefore its acceleration is $\omega \frac{d\sigma}{dt}$. To obtain the accurate equation of moments about C we must apply the effective force $\Sigma m \, . \, \omega \frac{d\sigma}{dt}$ in the reversed direction at the centre of gravity. But in small oscillations ω and $\frac{d\sigma}{dt}$ are both small quantities whose squares and products are to be neglected. Hence the moment of this force must be neglected, and the equation of motion will be the same as if C had been a fixed point.

It is to be observed that we may take moments about any point very near to the instantaneous centre of rotation, but it will usually be most convenient to take moments about the centre in its disturbed position. If there be any unknown reactions at the centre of rotation, their moments will then be zero.

If the accurate equation of moments about the instantaneous centre be required, the value of $\omega \frac{d\sigma}{dt}$ must be found from the peculiar circumstances of the problem under consideration. For example, if a body roll on a curve, then the arc $d\sigma$ is described by C when the body has turned through an angle $\frac{d\sigma}{\rho} + \frac{d\sigma}{\rho'}$ where ρ, ρ' are the radii of curvature of the body and curve at the point of contact, the curvatures being supposed to be in opposite directions. Hence, since ωdt is the angle turned through by the body in the time dt,

$$\frac{d\sigma}{dt} = \frac{\omega}{\frac{1}{\rho} + \frac{1}{\rho'}}.$$

68. If there be several bodies forming together a dynamic system, since the equation (B) is true for each of them separately, it will also be true when they are all taken together. In this case we cannot usually find a point C fixed with respect to all of them, and we shall therefore require

the following proposition to enable us to write down at once the value of

$$\Sigma m \frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right).$$

69. Prop. V. *The sum of the moments of all the effective forces of any rigid body about a point C is equal to the sum of their moments about the centre of gravity plus the moment about C of the effective force of the centre of gravity.* This proposition expressed in Algebraic language is

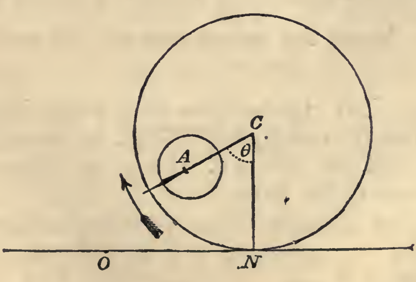
$$\Sigma m \frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) = Mk^2 . \frac{d\omega}{dt} + M . \frac{d}{dt}\left(\bar{r}^2 \frac{d\bar{\theta}}{dt}\right),$$

where ω is the angular velocity of the body, $\bar{r}$, $\bar{\theta}$, the co-ordinates of the centre of gravity referred to C as origin and any fixed direction through C as initial line.

Since $r^2 \dfrac{d\theta}{dt} = x \dfrac{dy}{dt} - y \dfrac{dx}{dt}$, the proposition follows at once from Art. 5.

70. Ex. *A sphere has a spherical eccentric cavity filled with water and rolls on a perfectly rough horizontal plane. To find the motion.*

Let M and m be the masses of the sphere and water,



a and b the radii of the two spherical surfaces, C and A their two centres. Let $CA = c$.

Let $x = ON$ be the abscissa of C and let $\theta =$ the angle NCA through which the sphere has turned.

First. Let the sphere and water be of the same density. Then C is the common centre of gravity of the whole system, and we may take moments about it as about a fixed point. Since the sides of the spherical cavity are supposed to be smooth, the water supposed to be originally at rest will have no rotation, hence the quantity represented by ω in Art. 69 is here nothing.

The equations are therefore

$$Mk^2 \frac{d^2\theta}{dt^2} + mc^2 \frac{d^2\theta}{dt^2} = Fa \quad\text{................ (1)},$$

$$(M+m)\frac{d^2x}{dt^2} = -F \quad\text{.............. (2)},$$

and the geometrical equation is

$$\frac{dx}{dt} = a\frac{d\theta}{dt} \quad\text{.............. (3)},$$

where $Mk^2 = (M+m)\frac{2}{5}a^2 - m\left(c^2 + \frac{2}{5}b^2\right)$.

Solving these equations by the first method, we have $F = 0$ and $\frac{d^2\theta}{dt^2} = 0$: hence the sphere moves with uniform velocity.

Secondly. Let the sphere and water be of different densities. Let G, which lies in CA, be the common centre of gravity, and let $CG = h$. Then by Prop. IV. taking moments about C, we have

$$Mk^2 \frac{d^2\theta}{dt^2} + mc^2 \frac{d^2\theta}{dt^2} = Fa + \frac{d^2x}{dt^2} h \cos\theta\,(M+m),$$

$$- (M+m)\,gh\sin\theta,$$

also

$$(M+m)\,\frac{d^2(x-h\sin\theta)}{dt^2} = -F,$$

and the geometrical equation is

$$\frac{dx}{dt} = a\,\frac{d\theta}{dt}.$$

Eliminating F and x from these equations, we get

$$\{Mk^2 + mc^2 + (M+m)\,a^2 - 2\,(M+m)\,ah\cos\theta\}\,\frac{d\,(\omega^2)}{d\theta}$$

$$+2\,(M+m)\,ah\sin\theta\,.\,\omega^2 = -2\,(M+m)\,gh\sin\theta, \text{ where } \frac{d\theta}{dt} = \omega.$$

Integrating both sides, we have

$$\left(\frac{Mk^2+mc^2}{M+m} + a^2 - 2\,ah\cos\theta\right)\omega^2 = C + 2\,(M+m)\,gh\cos\theta.$$

This equation gives the velocity of the sphere.

In this investigation it has been supposed that there has been no *jumping*. When the fluid has the same density as the sphere the common centre of gravity has no vertical motion, and the pressure on the table being $= (M+m)\,g$, is always positive, and thus there can be no jumping. But when the centre of gravity is not in the centre of the sphere then

$$\bar{y} = a - h\cos\theta,$$

$$\frac{d^2\bar{y}}{dt^2} = \frac{R}{M+m} - g,$$

whence R can be found in terms of θ. If the motion be very slow it is evident from the second equation that R cannot be negative; there will then be no jumping.

If R vanish and become negative, the sphere will leave the table and the centre of gravity will describe a parabola.

SECT. III. *Small Oscillations and Initial Motions.*

71. If a body be placed in a position of stable equilibrium and be then slightly disturbed it will make small oscillations about that position of rest. Suppose the disturbance originally given to the body to decrease without limit, then the consequent motion will also decrease without limit. But it will be found that the time of oscillation has in general a *finite limit.* This finite limit when it exists is called the time of a small oscillation. The smaller the motions, the more nearly does the time of oscillation become equal to this limit. When the motion is sufficiently small we may take this as the true time of oscillation.

Since the motion is supposed ultimately to vanish, it is obvious that we may neglect all squares and products of small quantities. Thus the equations will be greatly simplified. If we wish to preserve the linearity of our equations we must treat the equations in a different manner from that described in the previous section. For the chief method there described consists in multiplying both sides of the equations by quantities which in this case are very small, and we should then be obliged to consider terms of the second order. This is not usually found convenient in practice.

72. When the system admits of only one independent motion our object in general is to reduce the equations to the form

$$\frac{d^2s}{dt^2} + ns = a.$$

This is effected by neglecting the squares of the small quantity s. The solution of this equation is known to be

$$s = \frac{a}{n} + A \cos (\sqrt{n} \, . \, t + B),$$

where A and B are two arbitrary constants. The physical interpretation of this equation has been determined in treatises on Dynamics of a Particle. It is proved that it represents an oscillatory motion, that the period of a complete oscillation is $\frac{2\pi}{\sqrt{n}}$, and that the central point is at a distance from the origin equal to $\frac{a}{n}$. The extent of the oscillation is equal to A and depends on the initial conditions of the motion. But in this case the motion is supposed ultimately to vanish. Thus it appears that we may determine all we require directly from the differential equation and without considering the initial conditions of motion.

If n be negative the solution of the equation

$$\frac{d^2s}{dt^2} - ns = a$$

is
$$s = \frac{a}{-n} + Ae^{\sqrt{n}t} + Be^{-\sqrt{n}t}.$$

The motion will not be oscillatory since s will continually increase or decrease with t. If n be zero, we shall have to consider the terms of the second order in the differential equation.

73. *First Method.* When the system consists of a single body there is a very simple method of finding the motion, that is sometimes applicable.

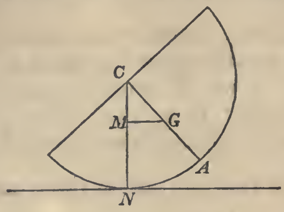
Find the instantaneous centre. This can be always effected when the directions of motion of two points are known. For if we draw from these points perpendiculars to their directions of motion, these perpendiculars will meet in the instantaneous centre of rotation.

It has been shewn in the preceding section that if we neglect the squares of small quantities we may take moments about the instantaneous centre as a fixed centre. Now as the unknown reactions will usually act at this point, their moments will be zero, and thus we shall in general have an equation containing only known quantities.

74. Ex. 1. *A hemisphere performs small oscillations on a perfectly rough horizontal plane: find the motion.*

Let C be the centre, G the centre of gravity of the hemisphere, N the point of contact with the rough plane.

Let the radius $= a$, $CG = c$, $\theta = \angle NCG$.



Here the point N is the centre of instantaneous rotation, because the plane being perfectly rough, sufficient friction will be called into play to keep N at rest.

Hence taking moments about N

$$(k^2 + GN^2)\frac{d^2\theta}{dt^2} = -gc \,.\, \sin\theta.$$

Since we can put $GN = a - c$ in the small terms, this reduces to

$$\{k^2 + (a-c)\}^2 \frac{d^2\theta}{dt^2} + gc \,.\, \theta = 0.$$

Therefore the time of a small oscillation is

$$= 2\pi\sqrt{\frac{k^2 + (a-c)^2}{cg}}.$$

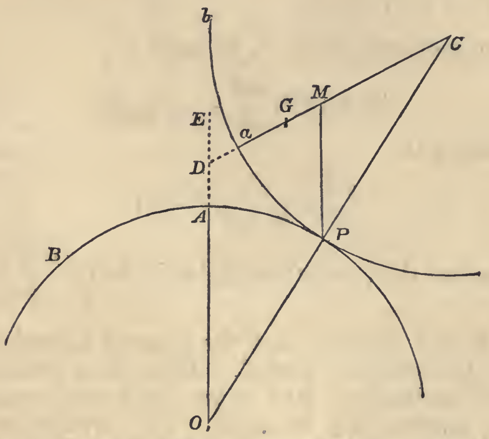
It is clear that $k^2 + c^2 =$ (rad.)^2 of gyration about C

$$= \frac{2}{5}a^2 \text{ and } c = \frac{3}{8}a.$$

If the plane had been smooth, M would have been the instantaneous axis, GM being the perpendicular on CN. For the motion of N is in a horizontal direction because the sphere remains in contact with the plane, and the motion of G is vertical by Art. 33. Hence the two perpendiculars GM, NM meet in the instantaneous axis. By reasoning similar to the above the time will be found to be $2\pi\sqrt{\frac{k^2}{cg}}$.

75. Ex. 2. *A cylindrical surface of any form rests in stable equilibrium on another perfectly rough cylindrical surface, the axes of the cylinders being parallel. A small disturbance being given to the upper surface, to find the time of a small oscillation.*

Let BAP and baP be sections of the cylinders perpendicular to their axes. Let OA, Ca be those normals to the two cylinders which before disturbance were vertical. Let OPC be the common normal at the time t. Through P draw a vertical line cutting aC in M, and let G be the centre of



gravity of the body. Then unless G be on the left-hand side of M, the body will be in unstable equilibrium, and there can be no oscillation.

Now we have only to determine the time of oscillation when the motion decreases without limit. Hence the arc aP will be ultimately zero, and therefore C and O may be taken as the centres of curvature of aP and AP. Let $r = OA$, $r' = Ca$, $c = aM$, and let θ be the angle which Ca makes with the vertical;

$$\therefore \theta = \angle CDE = \angle POA + \angle PCa$$

$$= \frac{\text{arc } PA}{r} + \frac{\text{arc } Pa}{r'} \text{ ultimately;}$$

also $$\theta = \angle\, aMP = \frac{\text{arc } Pa}{c} \text{ ultimately.}$$

But since one body rolls on the other,

$$\text{the arc } Pa = \text{arc } PA;$$

therefore equating the two values of θ,

$$\frac{1}{c} = \frac{1}{r} + \frac{1}{r'};$$

this equation determines c; also let $aG = c'$ and GM which is equal to $c - c' = c''$.

Taking moments about P, we have

$$\{k^2 + PG^2\} \frac{d^2\theta}{dt^2} = -c''g \,.\, \theta,$$

which reduces to

$$\frac{d^2\theta}{dt^2} + \frac{c''g}{k^2 + c''^2}\,\theta = 0;$$

therefore the time of oscillation is $2\pi\sqrt{\dfrac{k^2 + c''^2}{gc''}}$.

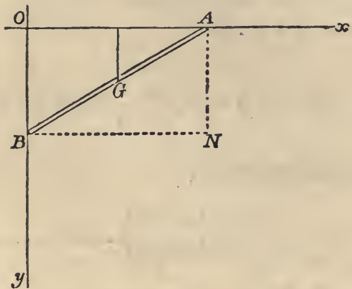
76. *Second Method.* Let the general Equations of motion of all the bodies be formed. Then if the positions about which the oscillations take place be known some of the quantities involved will be small. The squares and higher powers of these may be neglected, and then all the equations will become linear. If the unknown reactions be then eliminated, the resulting equations may easily be solved.

If the positions about which the oscillations take place be unknown it is not necessary to solve the Statical Problem first. We may by one process determine the positions of rest, ascertain whether they are stable or not, and find the time of oscillation. The method of proceeding will be best explained by an example.

77. Ex. *The ends of a uniform heavy rod AB of length $2l$ are constrained to move, the one along a horizontal line Ox, the other along a vertical line Oy. If the whole*

system turn round Oy with a uniform angular velocity ω, it is required to find the positions of equilibrium and the time of a small oscillation.

Let x, y be the co-ordinates of G the middle point of the



rod, θ the angle OAB the rod makes with Ox. Let R, R' be the reactions at A and B.

The effect of the rotation is the same as if the rod were at rest and each element dr of the rod were acted on by a force $\omega^2(x + r\cos\theta)\,dr$ tending from Oy, the distance r being measured from G towards A. All these forces are equivalent to a single force acting at G

$$= \int_{-l}^{+l} \omega^2 (x + r\cos\theta)\,dr = \omega^2 . x . 2l,$$

and a couple round G*

$$= \int_{-l}^{+l} \omega^2 (x + r\cos\theta)\, r\sin\theta\,dr = \omega^2 . 2l . \frac{l^2}{3} . \sin\theta\cos\theta.$$

* If a body in one plane be turning about an axis in its own plane with an angular velocity ω, a general expression can be found for the resultants of the centrifugal forces on all the elements of the body. Take the centre of gravity G as origin and the axis of y parallel to the fixed axis. Let c be the distance of G from the axis of rotation. Then all the centrifugal forces are equivalent to a single resultant force at

$$G = \int \omega^2 (c + x)\,dm$$
$$= \omega^2 . Mc, \text{ since } \bar{x} = 0,$$

and to a single resultant couple

$$= \int \omega^2 (c + x)\,y\,dm,$$
$$= \omega^2 \int xy\,dm \text{ since } \bar{y} = 0,$$
$$= \omega^2 . \text{ Product of inertia about } Gx,\ Gy.$$

Then we have the *dynamical equations*

$$\left.\begin{aligned} 2l\,.\frac{d^2x}{dt^2} &= -R' + \omega^2 x\,.\,2l \\ 2l\,.\frac{d^2y}{dt^2} &= -R + g\,.\,2l \\ 2l\,.\,k^2.\frac{d^2\theta}{dt^2} &= Rx - R'y - \omega^2.\,2l\,.\frac{l^2}{3}\sin\theta\cos\theta \end{aligned}\right\} \text{.........(1).}$$

and the *geometrical equations*

$$\left.\begin{aligned} x &= l\cos\theta \\ y &= l\sin\theta \end{aligned}\right\} \text{........................ (2).}$$

Eliminating R, R', from the equations (1), we get

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} + k^2\frac{d^2\theta}{dt^2} = gx - \omega^2 xy - \omega^2\frac{l^2}{3}\sin\theta\cos\theta \ldots (3).$$

To find the position of rest.

We observe that if the rod were placed in that position it would always remain there, and that therefore

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = 0, \quad \frac{d^2\theta}{dt^2} = 0.$$

This gives

$$gx - \omega^2 xy - \omega^2\frac{l^2}{3}\sin\theta\cos\theta = 0 \text{ .... (4).}$$

Joining this with equations (2), we get $\theta = \frac{\pi}{2}$, or $\sin\theta = \frac{3g}{4\omega^2 l}$, and thus the positions of equilibrium are found. Let any one of these positions be represented by

$$\theta = \alpha, \quad x = a, \quad y = b.$$

To find the motion of oscillation.

Let $x = a + x'$, $y = b + y'$, $\theta = \alpha + \theta'$, where x', y', θ'

are all small quantities, then we must substitute these values in equation (3). On the left-hand side since $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2\theta}{dt^2}$, are all small, we have simply to write a, b, α, for x, y, θ. On the right-hand side the substitution should be made by Taylor's Theorem, thus

$$f(a+x',\ b+y',\ \alpha+\theta') = \frac{df}{da}x' + \frac{df}{db}y' + \frac{df}{d\alpha}\theta'.$$

We know that the first term $f(a, b, \alpha)$ will be equal to nothing, because this was the very equation (4) from which a, b, α were found. We therefore get

$$a\frac{d^2y'}{dt^2} - b\frac{d^2x'}{dt^2} + k^2\frac{d^2\theta'}{dt^2} = (g-\omega^2 b)\,x' - \omega^2 ay' - \omega^2\frac{l^2}{3}\cos 2\alpha\,.\,\theta'.$$

But by putting $\theta = \alpha + \theta'$ in equations (2), we get by Taylor's Theorem

$$x' = -l\sin\alpha\,.\,\theta', \qquad y' = l\cos\alpha\,.\,\theta'.$$

Hence the equation to determine the motion is

$$(l^2+k^2)\frac{d^2\theta'}{dt^2} + \left(gl\sin\alpha + \frac{4}{3}\omega^2 l^2\cos 2\alpha\right)\theta' = 0.$$

Now, if $gl\sin\alpha + \frac{4}{3}\omega^2 l^2 \cos 2\alpha = n$ be positive when either of the two values of α is substituted, that position of equilibrium is *stable*, and the time of a small oscillation is

$$2\pi\sqrt{\frac{l^2+k^2}{n}}.$$

If n be negative the equilibrium is *unstable*, and there can be no oscillation.

If $n=0$, the body is in a position of *neutral* equilibrium, and we must calculate both sides of the equation as far as terms of the second order.

By a well-known transformation we have

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = \frac{d}{dt}\left(l^2\frac{d\theta}{dt}\right).$$

Hence the left-hand side of equation (3) becomes

$$(l^2+k^2)\,\frac{d^2\theta}{dt^2}.$$

The right-hand side becomes by Taylor's Theorem

$$-\frac{d}{d\alpha}\left(gl\sin\alpha+\frac{4}{3}\,\omega^2 l^2\cos 2\alpha\right)\frac{\theta'^2}{2}.$$

Hence the equation of motion true to the second order is

$$(l^2+k^2)\,\frac{d^2\theta'}{dt^2}=-\,C\frac{\theta'^2}{2},$$

where $C=gl\cos\alpha-\frac{8}{3}\,\omega^2l^2\sin 2\alpha$.

The equilibrium is unstable for a displacement in one direction and stable for a displacement in the opposite direction. Let C be positive and let α be the initial value of θ', then the time T of reaching the position of equilibrium is

$$T=\sqrt{\frac{3\,(l^2+k^2)}{C}}\int_\alpha^0\frac{d\theta'}{\sqrt{\alpha^3-\theta'^3}},$$

put $\theta'=\alpha\phi$, then

$$T=\sqrt{\frac{3\,(l^2+k^2)}{C}}\cdot\int_1^0\frac{d\phi}{\sqrt{1-\phi^3}}\cdot\frac{1}{\sqrt{\alpha}};$$

hence the time of reaching the position of rest varies inversely as the square root of the arc. Hence when the arc becomes ultimately zero, the time becomes infinite.

78. This problem might have been easily solved by the first method. For if the two perpendiculars to Ox, Oy at A and B meet in N, N is the instantaneous axis. Taking moments about N, we have the equation

$$\begin{aligned}(l^2+k^2)\,\frac{d^2\theta}{dt^2}&=gl\cos\theta-\int_{-l}^{+l}\omega^2\,(l+r)^2\sin\theta\cos\theta\,\frac{dr}{2l}\\&=gl\cos\theta-\frac{4l^2}{3}\,.\sin\theta\cos\theta\\&=f(\theta).\end{aligned}$$

Then the positions of equilibrium can be found from the equation

$$f(\alpha) = 0,$$

and the time of oscillation from the equation

$$(l^2 + k^2) \frac{d^2\theta'}{dt^2} = \frac{df(\alpha)}{d\alpha} . \theta'.$$

79. *Third Method.* When there are several bodies which may move independently of each other the number of equations and unknown reactions may be very great. In the second method we begin our process by eliminating from the equations all the unknown reactions, and no further use is made of the Dynamical Equations. We shall now explain a method of obtaining at once the result of this elimination, which will not require us to write down the primitive Dynamical Equations.

If we reverse the effective forces, by D'Alembert's Principle they will be in equilibrium with the impressed forces. Now applying the Principle of Virtual Velocities, we have by Todhunter's Statics, Art. 254,

$$\Sigma m \left(\frac{d^2x}{dt^2} \delta x + \frac{d^2y}{dt^2} \delta y\right) = \Sigma m (X\delta x + Y\delta y),$$

where δx δy are any small *arbitrary* displacements consistent with the geometrical relations, and X, Y are the resolved parts of the impressed forces, omitting all the reactions.

By referring to Art. 259 of Todhunter's Statics, it will be seen that this equation subdivides into as many equations as there are independent motions in the system. These with the geometrical equations will be sufficient to determine the motion. Suppose for example the system admit of only one independent motion, then x, y, &c. may be expressed in terms of some one variable, say θ. Let

$$x = f(\theta), \qquad y = \psi(\theta);$$

$$\therefore \delta x = f'(\theta)\, \delta\theta, \qquad \delta y = \psi'(\theta)\, \delta\theta;$$

then after substitution, $\delta\theta$ will divide out of the equation, and we shall have a Dynamical Equation free from all the unknown reactions.

80. If any of the bodies be a rigid body, the Σ on the

left-hand side becomes an integral, and the following proposition will be necessary.

Let $\bar{x}$, $\bar{y}$, be the co-ordinates of the centre of gravity; ϕ the angle any straight line makes with the axis of x, M the mass of the body. Then

$$\int dm\left(\frac{d^2x}{dt^2}\,\delta x+\frac{d^2y}{dt^2}\,\delta y\right)=M\left(\frac{d^2\bar{x}}{dt^2}\,\delta\bar{x}+\frac{d^2\bar{y}}{dt^2}\,\delta\bar{y}\right)$$
$$+Mk^2\frac{d^2\phi}{dt^2}\,\delta\phi:$$

or the virtual moment of all the effective forces is equal to the virtual moment of the whole mass collected at its centre of gravity, plus the virtual moment due to rotation round the centre of gravity.

This may be proved as follows: Let $x=\bar{x}+x'$, and $y=\bar{y}+y'$. Then, by Art. 5, the expression on the left-hand side becomes

$$M\left(\frac{d^2\bar{x}}{dt^2}\,\delta\bar{x}+\frac{d^2\bar{y}}{dt^2}\,\delta y\right)+\int dm\left(\frac{d^2x'}{dt^2}\,\delta x'+\frac{d^2y'}{dt^2}\,\delta y'\right).$$

Putting $x'=r\cos\phi$, $y'=r\sin\phi$, where r is independent of t, we get

$$\left.\begin{aligned}\frac{d^2x'}{dt^2}&=-r\sin\phi\,\frac{d^2\phi}{dt^2}-r\cos\phi\left(\frac{d\phi}{dt}\right)^2\\ \frac{d^2y'}{dt^2}&=+r\cos\phi\,\frac{d^2\phi}{dt^2}-r\sin\phi\left(\frac{d\phi}{dt}\right)^2\end{aligned}\right\}.$$

Multiplying these respectively by

$$\left.\begin{aligned}\delta x'&=-r\sin\phi\,\delta\phi\\ \delta y'&=+r\cos\phi\,\delta\phi\end{aligned}\right\},$$

the last term of the above expression becomes

$$\int dm\,r^2\frac{d^2\phi}{dt^2}\,\delta\phi=Mk^2\frac{d^2\phi}{dt^2}\,\delta\phi.$$

81. Ex. 1. Let us take the problem discussed in the second method.

The only impressed forces are $2lg$ acting at G, and the centrifugal force $\omega^2 (x + r \cos\theta)\, dr$ acting on each element dr and tending from Oy. The virtual moment of the former is $2lg\delta y$. The virtual moment of the latter is

$$= \omega^2 \int_{-l}^{+} (l + r) \cos\theta\, dr \,.\, \delta (l + r) \cos\theta$$

$$= -2l \,.\, \frac{4l^2}{3} \,.\, \omega^2 \cos\theta \,.\, \sin\theta \,.\, \delta\theta.$$

Hence the dynamical equation is

$$\frac{d^2x}{dt^2}\,\delta x + \frac{d^2y}{dt^2}\,\delta y + k^2\,\frac{d^2\theta}{dt^2}\,\delta\theta = g\delta y - \frac{4l^2}{3}\,\omega^2 \sin\theta \cos\theta\,\delta\theta.$$

The geometrical equations are

$$x = l\cos\theta, \quad y = l\sin\theta;$$

$$\therefore\ \delta x = -l\sin\theta\,\delta\theta, \quad \delta y = l\cos\theta\,\delta\theta,$$

substituting, we get

$$-\frac{d^2x}{dt^2}\,l\sin\theta + \frac{d^2y}{dt^2}\,l\cos\theta + k^2\frac{d^2\theta}{dt^2} = gl\cos\theta - \frac{4l^2}{3}\,\omega^2\sin\theta\cos\theta,$$

the very same equation which we obtained before.

The remainder of the solution is therefore the same as before.

82. Ex. 2. *Two rods AB, BC are connected by a smooth hinge at B, and are suspended from a fixed point by one extremity A. To determine the small oscillations of the system.*

Let AB, BC, make small angles θ, θ' with the vertical. Let x, y; x', y', be the co-ord. of their centres of gravity, x being measured downwards from the point of suspension. Let $2l$, $2l'$ be the lengths of the rods, $2lm$, $2l'm$ their masses, k, k' their radii of gyration about their respective centres of gravity.

The equation of motion is

$$l\left(\frac{d^2x}{dt^2}\,\delta x + \frac{d^2y}{dt^2}\,\delta y + k^2\,\frac{d^2\theta}{dt^2}\,\delta\theta\right)$$

$$+\,l'\left(\frac{d^2x'}{dt^2}\,\delta x' + \frac{d^2y'}{dt^2}\,\delta y' + k'^2\,\frac{d^2\theta'}{dt^2}\,\delta\theta'\right) = gl\delta x + gl'\,\delta x'.$$

The geometrical equations are

$$x = l\cos\theta, \qquad y = l\sin\theta,$$

$$x' = 2l\cos\theta + l'\cos\theta', \quad y' = 2l\sin\theta + l'\sin\theta'.$$

We have first to substitute for δx, δy, &c. in the equation of motion. As that equation will be subsequently divided by $\delta\theta$ or $\delta\theta'$, it must be obtained in the first instance correct to the second order. The terms on the left-hand side contain $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, &c. and we may substitute for δx, δy, &c. their approximate values obtained by taking only the terms of *the first order;* but on the right-hand side we must substitute the values of δx, &c. correct to *the second order.*

From the geometrical equations, we have

$$\delta x = -\,l\theta\,\delta\theta, \qquad \delta y = l\delta\theta,$$

$$\delta x' = -\,2l\theta\,\delta\theta - l'\theta'\,\delta\theta', \quad \delta y' = 2l\,\delta\theta + l'\delta\theta';$$

on the left-hand side of the equation of motion, we may put

$$\delta x = 0, \quad \delta x' = 0.$$

Hence, substituting, we get

$$\left.\begin{aligned}&\left(l\,\frac{d^2y}{dt^2} + 2l'\,\frac{d^2y'}{dt^2} + k^2\,\frac{d^2\theta}{dt^2}\right) l\delta\theta\\ +&\left(l'\,\frac{d^2y'}{dt^2} + k^2\,\frac{d^2\theta'}{dt^2}\right) l'\delta\theta'\end{aligned}\right\} = \begin{cases}-gl\theta\,(l+2l')\,\delta\theta\\ -gl'^2\theta'\delta\theta'.\end{cases}$$

But since $\delta\theta$ and $\delta\theta'$ are independent, this gives

$$\left.\begin{aligned} l\frac{d^2y}{dt^2}+2l'\frac{d^2y'}{dt^2}+k^2\frac{d^2\theta}{dt^2} &= -g(l+2l')\,\theta \\ l'\frac{d^2y'}{dt^2}+k'^2\frac{d^2\theta'}{dt^2} &= -gl'\theta' \end{aligned}\right\}.$$

Substituting in these equations for y and y', the approximate values

$$y=l\theta,\quad y'=2l\theta+l'\theta',$$

we get

$$\left.\begin{aligned} (4ll'+l^2+k^2)\frac{d^2\theta}{dt^2}+2l'^2\frac{d^2\theta'}{dt^2} &= -g(l+2l')\,\theta \\ 2ll'\frac{d^2\theta}{dt^2}+(l'^2+k'^2)\frac{d^2\theta'}{dt^2} &= -gl'\theta' \end{aligned}\right\}.$$

To solve these, assume

$$\theta=A\sin(nt+\alpha),$$
$$\theta'=A'\sin(nt+\alpha),$$

substituting, we have

$$\left.\begin{aligned} \{(4ll'+l^2+k^2)A+2l'^2A'\}\,n^2 &= g(l+2l')\,A \\ \{2ll'.A+(l'^2+k'^2)A'\}\,n^2 &= gl'A' \end{aligned}\right\}.$$

Eliminating, we have

$$\{(4ll'+l^2+k^2)\,n^2-g(l+2l')\}\,\{(l'^2+k'^2)\,n^2-gl'\}=4ll'^3n^4,$$

$$\frac{A'}{A}=\frac{-2ll'n^2}{(l'^2+k'^2)\,n^2-gl'}.$$

The first equation is a quadratic to determine n^2; it is easily seen that both its roots are positive. Let the four values of n thus obtained be $\pm n_1$ and $\pm n_2$. Then the oscillation is represented by the equation

$$\theta=A_1\sin(n_1t+\alpha_1)+A_2\sin(n_2t+\alpha_2),$$
$$\theta'=A_1'\sin(n_1t+\alpha_1)+A_2'\sin(n_2t+\alpha_2).$$

The four arbitrary constants A_1, A_2, α_1, α_2, are to be determined by the initial values of θ, θ', $\frac{d\theta}{dt}$, $\frac{d\theta'}{dt}$. The negative values of n give only the same terms over again.

Thus the motion consists of two oscillations whose periods are $\frac{2\pi}{\sqrt{n_1}}$ and $\frac{2\pi}{\sqrt{n_2}}$. These go on together and do not interfere in any way with each other. If there had been three rods we should have had three oscillations, and so on.

Initial Motions.

83. PROP. *A system of bodies being in equilibrium, one of the supports suddenly gives way. It is required to find the reactions on the other points of support.*

Suppose first that the system consists only of a single body, and let it receive any small displacement. Let x, y be the displacements of the centre of gravity along the axes, θ the angle turned through.

In the beginning of the motion, x, y, and θ, are very small, and hence their squares and higher powers may be rejected. The geometrical equations will therefore take the form

$$Ax + By + C\theta = D,$$

where A, B, C, D are some constants.

The geometrical equations must be found from the *displaced* position of the body, because we require to differentiate them; but this is not the case with the dynamical equations. These we write down at the instant when the body begins to move. Thus we have

$$\frac{d^2x}{dt^2} = \text{function of reactions and known quantities,}$$

$$\frac{d^2y}{dt^2} = \text{...}$$

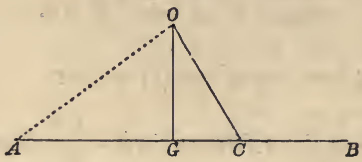
$$k^2\frac{d^2\theta}{dt^2} = \text{...}$$

By differentiating the geometrical equations and substituting from the dynamical equations we obtain sufficient equations to determine the *initial* values of the reactions.

If there be more than one body in the system, the process is exactly similar.

84. Ex. *A circular disc is hung up by three equal strings attached to three points at equal distances in its circumference, and fastened to a peg vertically over the centre of the disc. One of these strings is suddenly cut. To determine the initial tension of the other two.*

Let O be the peg, AB the circle seen by an eye situated



in its plane. Let OA be the string which is cut and C the centre of the chord joining the other two strings. Let G be the initial position of the centre of gravity.

Let $2\alpha =$ the angle between two strings, $l =$ the length of each string, $a =$ the radius of the disc. Let x, y be the coordinates of the displaced position of the centre of gravity with reference to G, and let θ be the angle the displaced position of the disc makes with AB.

Then the equations of motion are

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= 2T\cos\alpha\cos\beta \\ m\frac{d^2y}{dt^2} &= mg - 2T\cos\alpha.\sin\beta \\ mk^2\frac{d^2\theta}{dt^2} &= 2T\cos\alpha.c.\sin\beta \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(1),$$

where β is the known angle OCG and $c = GC$.

These are the equations of motion at the instant when the system begins to move. The geometrical equations are to be found from the displaced position.

The co-ordinates of C will be $x-c$ and $y-c\theta$. Let $OG=b$. Then since the length OC remains constant, we have

$$(x-c)^2+(y-c\theta+b)^2=b^2+c^2;$$

$$\therefore\ -2cx+2b(y-c\theta)=0,$$

by rejecting the squares of all small quantities. Differentiating we get

$$c\frac{d^2x}{dt^2}-b\frac{d^2y}{dt^2}+bc\frac{d^2\theta}{dt^2}=0.$$

Substituting from equations (1) we get

$$2Tc\cos\alpha\cos\beta-mgb+2Tb\cos\alpha\sin\beta-\frac{bc^2}{k^2}2T\cos\alpha\sin\beta=0,$$

which is an equation to determine T.

The tension T' before the string was cut is given by the equation

$$3T'\cos\gamma=mg,$$

where $\gamma=\angle AOG$. Thus the change of tension can be determined.

85. It is not absolutely necessary to express the geometrical equations in a linear form previous to differentiation. Supposing one of these equations to be

$$\phi(x, y\ldots)=0,$$

then, differentiating, we get

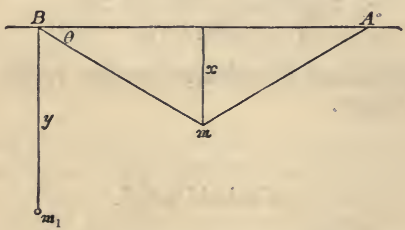
$$\phi'(x)\cdot\frac{dx}{dt}+\phi'(y)\frac{dy}{dt}+\ldots=0.$$

Differentiating again and remembering that the initial values of $\frac{dx}{dt}$, $\frac{dy}{dt}$ &c. are zero, since the system starts from rest, we get

$$\phi'(x)\frac{d^2x}{dt^2} + \phi'(y)\frac{d^2y}{dt^2} + \ldots = 0.$$

Substituting for $\frac{d^2x}{dt^2}$, $\frac{d\,y}{dt^2}$, &c. their values given by the dynamical equations, we have one of the equations required to determine the unknown tensions and reactions.

Ex. *A fine string attached to a fixed point A carries a small ring of mass m, and passing over a small pulley B in the same horizontal plane with the fixed point has a mass $m_1 + m_2$ attached to the free extremity. The system being in equilibrium, the mass m_2 is removed. Shew that the strain on the fixed point is instantly reduced by* $\frac{m_2(m_1+m_2)}{(m_1+m_2)^2+m_1 m}$ *times its former value.*



Let T_1 and T be the tensions of the string before and after the change. Let x and y be the distances of m and m_1 from the horizontal line AB, and let θ be the angle the part Bm of the string makes with AB.

The equations of motion are

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= mg - 2T\sin\theta \\ m_1\frac{d^2y}{dt^2} &= m_1 g - T \end{aligned}\right\} \text{initially} \ldots\ldots\ldots(1).$$

Also $$2\sqrt{(x^2+a^2)}+y=l,$$
where $AB=2a$ and l is the length of the string;

$$\therefore\ 2\frac{x}{\sqrt{(x^2+a^2)}}\frac{dx}{dt}+\frac{dy}{dt}=0;$$

$$\therefore\ 2\frac{x}{\sqrt{(x^2+a^2)}}\frac{d^2x}{dt^2}+\frac{d^2y}{dt^2}=0,\text{ initially} \quad\ldots\ldots\ldots\ldots(2).$$

Substituting, we get

$$2\,.\sin\theta\left(g-2\frac{T}{m}\sin\theta\right)+\left(g-\frac{T}{m_1}\right)=0;$$

$$\therefore\ T=mm_1g\frac{2\sin\theta+1}{4m_1\sin^2\theta+m}.$$

But since the system is at rest when we put m_1+m_2 for m_1,

$$2\sin\theta=\frac{m}{m_1+m_2};$$

$$\therefore\ \frac{T-T_1}{T_1}=\frac{m_2(m_1+m_2)}{(m_1+m_2)^2+m_1m}.$$

This problem and its solution are due to Mr C. B. Clarke of Queens' College.

EXAMPLES.

Sections I. and II.

1. A rod is capable of moving about one extremity upon a smooth horizontal plane: an elastic string is attached to the other extremity, and is made fast to the plane in such a manner that when the string has its natural length, the rod and string are in the same straight line; if the rod be drawn from the position in which the string has its natural length into any other and then let go, the angular velocity acquired in returning to its original position will be proportional to the initial extension of the string; Hooke's Law being supposed to hold throughout the motion.

2. A sphere rests on the top of a fixed sphere, and is very slightly displaced; determine where it will leave the fixed sphere, (1) when the surfaces are smooth, (2) when they are perfectly rough.

Result. Let θ be the angle the line joining the centres of the two spheres makes with the vertical at the moment of separation, then when the spheres are smooth, $\cos\theta = \frac{2}{3}$, when rough, $\cos\theta = \frac{10}{17}$.

3. A cylinder with a hemispherical end moves on a horizontal plane from a given position. Find in any position the angular velocity of the body, the velocity of its centre of gravity, and the pressure on the horizontal plane; (1) when the plane is perfectly smooth, (2) when perfectly rough.

4. A heavy uniform sphere rolls on a rough plane and is acted on by a fixed centre of force in the plane varying inversely as the square of the distance; if the sphere be projected along the plane from a given point in it, in a direction opposite to that of the centre of force, find the roughness of the plane at any point, supposing the whole of it to be required.

5. A perfectly rough cylinder is placed on an inclined plane, and an elastic band tight but unstretched and parallel to the inclined plane has one extremity fixed, while the other is attached to the cylinder exactly opposite to its line of contact with the inclined plane. All external support being removed motion ensues. Determine the velocity in any given position, and how far the cylinder will descend.

6. Two equal heavy spheres one solid and the other hollow, and the hollow filled with fluid, are revolving with the same angular velocity about a horizontal axis and are laid side by side on a rough horizontal plane, the coefficient of friction for both being μ; if the interior radius of the sphere be one-half of the exterior, and the density of the fluid equal to that of the solid, find the distance between them at any time, supposing that they move in parallel lines.

7. A smooth wire without inertia is bent into the form of a helix which is capable of revolving about a vertical axis coinciding with a generating line of the cylinder on which it is traced. A small heavy ring slides down the helix, starting from a point in which this vertical axis meets the helix: prove that the angular velocity of the helix will be a maximum when it has turned through an angle θ given by the equation

$$\cos^2\theta + \tan^2\alpha + \theta\sin 2\theta = 0,$$

α being the inclination of the helix to the horizon.

8. Two equal uniform rods of length $2a$, loosely jointed at one extremity, are placed symmetrically upon a fixed smooth sphere of radius $\frac{\sqrt{2a}}{3}$, and raised into a horizontal position so that the hinge is in contact with the sphere. If they be allowed to descend under the action of gravity, show that, when they are first at rest, they are inclined at an angle $\cos^{-1}\frac{1}{3}$ to the horizon, that the points of contact with the sphere are the centres of oscillation of the rods relatively to the hinge, that the pressure on the sphere at each point of contact equals one-fourth the weight of either rod, and that there is no strain on the hinge.

9. Two circular discs are on a smooth horizontal plane; one, whose radius is n times that of the other, is fixed, an elastic string wraps round them so that those portions of it not in contact with the discs are common interior tangents the natural length of the string being the sum of the circumferences. The moveable disc is drawn from the other till the tension of the string is T, prove that if it be now let go, the velocity acquired when it comes in contact with the fixed disc will be

$$\frac{T}{\lambda}\sqrt{\frac{2(n+1)\pi a \, . \, \lambda}{m}},$$

where m is the mass of the moving disc, λ the modulus of elasticity, a the radius of the moving disc.

10. Two masses m, m' are connected by an inextensible string, and laid over a double inclined plane of mass $m+m'$, which is capable of moving freely on a smooth horizontal plane. If the section of the inclined plane be isosceles, and α the inclination of its sides to the horizon, the system may be kept in a state of relative equilibrium by the force

$$2(m-m')\,g\tan\alpha$$

applied to the plane.

11. A body whose centre of gravity oscillates in a straight line under the action of a force, which tends to a fixed point, and varies as the distance, has an angular velocity communicated to it about a principal axis through its centre of gravity, which is perpendicular to the direction of motion; show that the instantaneous axis traces out in space an elliptic cylinder.

12. Two straight equal and uniform rods are connected at their ends by two strings of equal length a, so as to form a parallelogram. One rod is supported at its centre by a fixed axis about which it can turn freely, this axis being perpendicular to the plane of motion which is vertical. Show that the middle point of the lower rod will oscillate in the same way as a simple pendulum of length a, and that the *angular* motion of the rods is independent of this oscillation.

13. Three equal and perfectly smooth balls are in contact, each with the other two on a perfectly smooth plane, and another of the same size rests upon them. Supposing the motions of the balls to commence from these positions, find the velocity of each after the upper ball has descended through a given space.

14. A fine string is attached to two points A, B in the same horizontal plane, and carries a weight W at its middle point. A rod whose length is AB and weight W, has a

ring at either end, through which the string passes, and is let fall from the position AB. Show that the string must be at least $\frac{5}{3}AB$, in order that the weight may ever reach the rod.

Also if the system be in equilibrium, and the weight be slightly and vertically displaced, determine the time of its small oscillations.

15. Three equal particles A, B, C repelling each other with any forces, are tied together by three strings of unequal length, so as to form a triangle. If the string joining B and C be cut, prove that the instantaneous changes of tension of the strings joining BA, CA are $\frac{1}{2}T\cos B$ and $\frac{1}{2}T\cos C$ respectively, where B and C are the angles opposite the strings joining CA, AB respectively.

16. Three pieces of one uniform wire, rigidly connected so as to form a triangle ABC, are in motion; find the directions of the strains in the connections of the angles.

Result. The strain at A makes an angle

$$\tan^{-1}\left(\frac{\sin B - \sin C}{1 + \cos B + \cos C}\right)$$

with the side BC.

17. A fine thread is enclosed in a smooth circular tube which rotates freely about a vertical diameter; prove that, in the position of relative equilibrium, the inclination (θ) to the vertical, of the diameter through the centre of gravity of the thread, will be given by the equation

$$\cos\theta = \frac{g}{a\omega^2\cos\beta},$$

where ω is the angular velocity of the tube, a its radius, and $2a\beta$ the length of the thread. Explain the case in which the value of $a\omega^2\cos\beta$ lies between g and $-g$.

18. A spherical hollow is made in a cube of glass, and a particle is placed within. The cube is then set in motion on a smooth horizontal plane so that the particle just gets round the sphere, remaining in contact with it. Find the velocity of projection.

19. A perfectly rough ball is placed within a hollow cylindrical garden-roller at its lowest point, and the roller is then drawn along a level walk with a uniform velocity V. Show that the ball will roll quite round the interior of the roller, if

$$V^2 \text{ be} > \frac{20}{7} g (b-a),$$

a being the radius of the ball, and b of the roller.

20. A spherical shell (of radius a and mass m) rolls along a rough horizontal plane, whilst a smooth particle P oscillates within the shell in the vertical plane in which the centre of the shell moves, the particle being never very far from the lowest point. Show that the time of its oscillation will be the same as that of a simple pendulum of length

$$\frac{ma(a^2+k^2)}{(m+P)(a^2+mk^2)},$$

where k is the radius of gyration of the shell about a diameter.

21. A sphere with a hollow spherical eccentric cavity within it having a radius of the former for its diameter is placed on a perfectly rough inclined plane, with the centre of gravity at its shortest distance from the plane and left to itself: find the angular velocity of the body when it has just rolled once round, and the pressure on the plane.

22. A square formed of four similar uniform rods jointed freely at their extremities is laid upon a smooth horizontal table, one of its angular points being fixed: if angular velocities ω, ω' in the plane of the table be communicated to the

two sides containing this angle, show that the greatest value of the angle (2α) between them is given by the equation

$$\cos 2\alpha = -\frac{5}{6}\frac{(\omega - \omega')^2}{\omega^2 + \omega'^2}.$$

Section III.

23. Two equal and parallel cylinders are rigidly connected together by a straight rod. One cylinder makes small oscillations by rolling on a perfectly rough plane, the other cylinder being supported by the rod which passes through a slit in the plane. The time of oscillation being the same whichever cylinder is uppermost, prove that the length of the simple equivalent pendulum is equal to the distance between the cylinders.

24. A uniform rod of length $2c$ rests in stable equilibrium with its lower end at the vertex of a cycloid whose plane is vertical and axis downwards, and passes through a small smooth fixed ring situated in the axis at a distance b from the vertex. Show that if the equilibrium be slightly disturbed, the rod will perform small oscillations with its lower end on the arc of the cycloid in the time

$$2\pi\sqrt{\frac{a\{c^2 + 3(b-c)^2\}}{3g(b^2 - 4ac)}},$$

where $2a$ is the length of the axis of the cycloid.

25. Two rods are jointed at one end by a compass-joint, and the other ends slide by rings on a vertical smooth circle; find the condition, and the time of oscillation.

26. A small smooth ring slides on a circular wire of radius a which is constrained to revolve about a vertical axis in its own plane, at a distance c from the centre of the ring, with a uniform angular velocity $\sqrt{\frac{g\sqrt{2}}{c\sqrt[3]{2} + a}}$; show that the ring will be in a position of stable relative equilibrium when

the radius of the circular wire passing through it is inclined at an angle 45° to the horizon; and that if the ring be slightly displaced, it will perform a small oscillation in the time

$$2\pi \left\{\frac{a\sqrt{2}}{g} \cdot \frac{c\sqrt{2}+a}{c\sqrt{8}+a}\right\}^{\frac{1}{2}}.$$

27. Two equal extensible strings, stretched in a horizontal plane, have their ends fixed so as to touch at their middle points a circular disc at the extremities of a diameter. Supposing the strings to be nailed to the disc at these points, find the time of a small oscillation, (1) when the circle is turned through a small angle, the centre being unmoved, (2) when the centre is displaced in the line perpendicular to the strings.

Interpret your result supposing that in the position of equilibrium the string is unstretched.

28. Two points B, C of a circular ring, moveable in its own plane about its centre, are connected with a fixed point A by elastic strings, the natural length of each of which is equal to the shortest distance (c) between A and the ring; in the position of equilibrium AB, AC are tangents to the ring; supposing the ring turned through any angle, calculate the motion; and show that the time of a small oscillation is $\pi\sqrt{\frac{mc}{2\lambda}}$, where m is the mass of the ring and λ the modulus of elasticity of the strings.

29. A uniform bar suspended by two equal parallel strings from two points in the same horizontal line, is turned through a given angle about the vertical line through its middle point: find the angular velocity of the bar in its lowest position, and the time of a small oscillation when the initial displacement is small.

30. The upper extremity of a uniform beam, of length $2a$, is constrained to slide along a smooth horizontal rod without inertia, and the lower along a smooth vertical rod, through the upper extremity of which the horizontal rod

passes: the system rotates freely about the vertical rod: prove that if α be the inclination of the beam to the vertical when in a position of relative equilibrium, the angular velocity of the system will be $\left(\frac{3g}{4a\cos\alpha}\right)^{\frac{1}{2}}$: and if the beam be slightly displaced from this position, show that it will make small oscillations in the time

$$\frac{4\pi}{\left(\frac{3g}{a}(\sec\alpha + 3\cos\alpha)\right)^{\frac{1}{2}}}.$$

31. Six stretched elastic strings of the same material are attached to the angular points of a regular hexagon, the length of each stretched string being equal to that of a side of the hexagon, and they meet in a point to which a little insect of given mass clings,—while it is slightly displaced in the direction of one of the strings, having given the modulus of elasticity, find the number of oscillations per second, neglecting the attraction of gravity.

32. Four equal rods are connected by smooth joints at their extremities, so as to form a rhombus: a constant force mf is applied to each rod at its middle point, and perpendicular to its length, each force tending outwards. If the equilibrium of the system be slightly disturbed by pressing two opposite corners towards each other, and the system be then abandoned to the action of the forces, show that the time of a small oscillation in the *form* of the system is $= 2\pi\sqrt{\frac{2a}{3f}}$ where m = the mass, and a = the length of each rod.

33. A light uniform lamina in the form of a regular trapezoid is suspended by one of the parallel edges, and a weight Mg is uniformly distributed over the opposite edge; supposing the lamina to be elastic only in the direction of the breadth, find the position of equilibrium and the time of a small vertical oscillation.

If $2a$ and $2b$ be the lengths of the parallel edges, l the breadth of the lamina when unstretched,

$$T = 2\pi \sqrt{\frac{Ml(\log a - \log b)}{2E(a-b)}}.$$

34. A sphere whose centre of gravity is not in its centre is placed on a rough table; the coefficient of friction being μ, determine whether it will begin to slide or to roll.

35. An equilateral triangle is suspended from a point by three strings, each equal to one of the sides, attached to its angular points, if one string be cut, show that the tensions of the other two are diminished in the ratio of 36 : 43.

36. A horizontal rod of mass m and length $2a$, hangs by two parallel strings of length $2a$ attached to its ends: an angular velocity ω being suddenly communicated to it about a vertical axis through its centre, show that the initial increase of tension of either string equals $\frac{ma\omega^2}{4}$, and that the rod will rise through a space $\frac{a^2\omega^2}{6g}$.

37. A uniform solid, in the form of a paraboloid of revolution, rests with its vertex on a smooth horizontal plane. It is divided symmetrically by a vertical plane. Explain why the pressure on the plane is instantly diminished; find the change of pressure.

38. A circular ring is fixed in a vertical position upon a smooth horizontal plane, and a small ring is placed on the circle, and attached to the highest point by a string, which subtends an angle α at the centre; prove that if the string be cut and the circle left free, the pressures on the ring before and after the string is cut are in the ratio $M + m\sin^2\alpha : M$, m and M being the masses of the ring and circle.

39. Two uniform equal rods are placed in the form of the letter X on a smooth horizontal plane, the upper and lower extremities being connected by equal strings; shew that whichever string be cut, the tension of the other is the same function of the inclination of the rods, and initially is $\frac{3}{8}g\sin\alpha$, where α is the initial inclination of the rods.

CHAPTER V.

MOTION OF A RIGID BODY IN THREE DIMENSIONS.

SECT. I. *The Geometry of the Motion of a Rigid Body.*

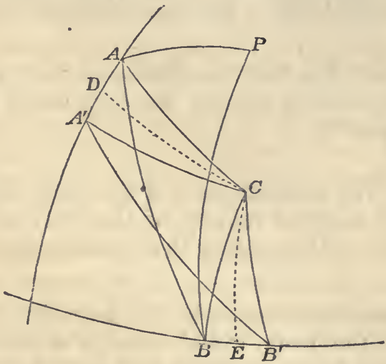
86. IF the particles of a body be rigidly connected, then whatever be the nature of the motion generated by the forces, there must be some general relations between the motions of the particles of the body. These must be such that if the motion of three points not in the same straight line be known, that of every other point may be deduced. It will then in the first place be our object to consider the general character of the motion of a rigid body apart from the forces that produce it, and to reduce the determination of the motion of every particle to as few independent quantities as possible: and in the second place we shall consider how when the forces are given these independent quantities may be found.

87. PROP. I. *One point of a moving rigid body being fixed, it is required to deduce the general relations between the motions of the other points of the body.*

Let O be the fixed point and let it be taken as the centre of a moveable sphere which we will suppose fixed in the body. Let the radius vector to any point Q of the body cut the sphere in P, then the motion of every point Q of the body will be represented by that of P.

If the displacements of two points A, B, on the sphere in the small time dt be given as AA', BB', then clearly the displacement of any other point P on the sphere may be found by constructing on $A'B'$ as base a triangle $A'P'B'$ similar and equal to APB. Then PP' will represent the displacement of P. It may be assumed as evident, or it may be proved as in Euclid, that on the same base and on the same side of it there cannot be two triangles on the same sphere, which have their sides terminated in one extremity of the base equal to one another, and likewise those terminated in the other extremity.

Let D and E be the middle points of the arcs AA', BB', and let DC, EC be arcs of great circles drawn perpendicular to AA', BB' respectively. Then clearly $CA = CA'$ and



$CB = CB'$, and therefore since the bases AB, $A'B'$ are equal, the two triangles ACB, $A'CB'$ are equal and similar. Hence the displacement of C is zero.

If we had taken any other points besides A and B to start with, we should still have obtained the same point C. For let P be any other point on the sphere. Then since the triangles APB, $A'P'B'$ are equal and similar, and also the triangles ACB, $A'CB'$, the same formula that determines CP will determine CP', hence $CP=CP'$. Therefore if PP' be bisected in R, RC will be perpendicular to PP'.

Also it is evident since the displacements of O and C are zero, that the displacement of every point in the straight line OC is also zero.

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Hence if a body be in motion in any manner about a fixed point, there is at every instant a straight line OC such that the displacement of every point in it in the time dt is zero, and there is only one such line.

88. It has been proved above if any point P be displaced to P', that $CP = CP'$. We shall now prove that the angle PCP' is the same wherever the point P is taken. For let A and B be any two points on the sphere: then since $CA = CA'$ and $CB = CB'$ and the base AB is equal to the base $A'B'$, the triangles are equal and the angle $ACB =$ the angle $A'CB'$. Removing the common part we have left the angle $ACA' =$ the angle BCB'. Let this constant angle be called $d\theta$.

It follows therefore that *the displacement of every particle P may be represented by turning the body round OC as axis through the angle $d\theta$.*

89. Def. The ultimate ratio of this angle $d\theta$ to the time dt is called the angular velocity of the body about OC, and the straight line OC is called the instantaneous axis at the time t.

The angular velocity may also be defined to be the angle through which the body would turn in a unit of time if it continued to turn throughout that unit with the same angular velocity which it had at the proposed instant, and about the same axis.

90. Prop. II. *To explain what is meant by a body having angular velocities about more than one axis at the same time.*

A body in motion is said to have an angular velocity ω about a straight line, when, the body being turned round this straight line through an angle ωdt, every point of the body is brought from its position at the time t to its position at the time $t + dt$.

Suppose that during three successive intervals each of time dt, the body is turned successively round three different straight lines OA, OB, OC meeting at a point O through angles $\omega_1 dt$, $\omega_2 dt$, $\omega_3 dt$. Then we shall first prove that the final position is the same in whatever order these rotations are effected. Let P be any point in the body, and let its distances from OA, OB, OC, respectively be r_1, r_2, r_3. First let the body be turned round OA, then P receives a displacement $\omega_1 r_1 dt$. By this motion let r_2 be increased to $r_2 + dr_2$, then the displacement caused by the rotation about OB will be in magnitude $\omega_2 (r_2 + dr_2)\, dt$. But according to the principles of the Differential Calculus we may in the limit neglect the quantities of the second order, and the displacement becomes $\omega_2 r_2 dt$. So also the displacement due to the remaining rotation will be $\omega_3 r_3 dt$. And these three results will be the same in whatever order the rotations take place. In a similar manner we can prove that the *directions* of these displacements will be independent of the order. The final displacement is the diagonal of the parallelopiped described on these three lines as sides, and is therefore independent of the order of the rotations. Since then the three rotations are quite independent, they may be said to take place simultaneously.

When a body is said to have angular velocities about three different axes it is only meant that the motion may be determined as follows. Divide the whole time into a number of small intervals each equal to dt. During each of these turn the body round the three axes successively through angles $\omega_1 dt$, $\omega_2 dt$, $\omega_3 dt$. Then when dt diminishes without limit the motion during the whole time will be accurately represented.

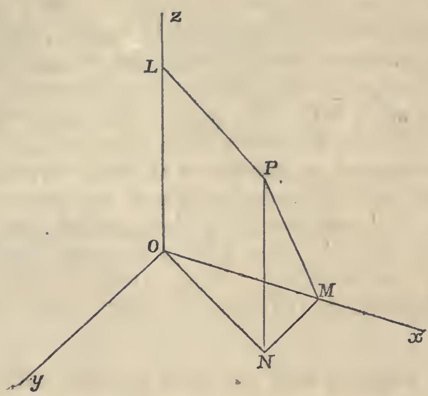
91. PROP. III. *Given the angular velocities ω_1, ω_2, ω_3 of a body about three axes Ox, Oy, Oz at right angles, to determine the actual velocities of a particle whose co-ordinates are x, y, z.*

These angular velocities are supposed positive when they tend the same way round the axes that positive couples tend in Statics. Thus the positive directions of ω_1, ω_2, ω_3 are respectively from y to z, from z to x, and from x to y.

Let us determine the velocity of P parallel to the axis of z. Let PN be the ordinate z, and let PM be drawn perpendicular to Ox. The velocity of P due to the rotation about Ox is clearly $\omega_1 PM$. Resolving this along NP we get

$$\omega_1 PM \sin NPM = \omega_1 y.$$

Similarly that due to the rotation about Oy is $-\omega_2 x$; and



that due to the rotation about Oz is 0. Hence the whole velocity of P parallel to Oz is

$$\frac{dz}{dt} = \omega_1 y - \omega_2 x,$$

and the velocities parallel to the other axes

$$\frac{dx}{dt} = \omega_2 z - \omega_3 y,$$

$$\frac{dy}{dt} = \omega_3 x - \omega_1 z.$$

92. The quantities ω_1, ω_2, ω_3, are called the angular velocities of the body about the axes of x, y, z respectively, but they must be carefully distinguished from the angular velocities of any particular particle of the body about the

same axes. Let P be any particle of the body whose co-ordinates are x, y, z, and draw $PL = r$ perpendicular to the axis of z. Let θ be the angle xON, then the instantaneous angular velocity of P about Oz is $\frac{d\theta}{dt}$.

But
$$r^2 \frac{d\theta}{dt} = x\frac{dy}{dt} - y\frac{dx}{dt}$$
$$= \omega_3 r^2 - xz \,.\, \omega_1 - yz\omega_2$$

by substituting for $\frac{dx}{dt}$, $\frac{dy}{dt}$, their values just found;

$$\therefore \frac{d\theta}{dt} = \omega_3 - \omega_1 \frac{xz}{r^2} - \omega_2 \frac{yz}{r^2}.$$

Hence the angular velocity of a particle about Oz is the same as that of the body when the particle lies in the plane of xy or when it lies in the plane given by

$$y = -x\frac{\omega_1}{\omega_2}.$$

93. Prop. IV. *Given the angular velocities ω_1, ω_2, ω_3, of a body about three axes Ox, Oy, Oz, at right angles, to determine the position of the instantaneous axis and the angular velocity about it.*

Since the velocity of every point in the instantaneous axis is zero, its position may be at once found by equating to nothing the expressions for $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, given in Prop. III. Thus we have

$$\left.\begin{aligned} 0 &= \omega_1 y - \omega_2 x \\ 0 &= \omega_2 z - \omega_3 y \\ 0 &= \omega_3 x - \omega_1 z \end{aligned}\right\},$$

and the equations to the instantaneous axis are therefore

$$\frac{x}{\omega_1} = \frac{y}{\omega_2} = \frac{z}{\omega_3}.$$

Let ω be the angular velocity about the instantaneous axis, and α, β, γ the angles it makes with the axes of co-ordinates. Let x, y, z be the co-ordinates of any point P in the instantaneous axis, and let N be the foot of the ordinate z.

Let us consider the velocity of the point N. Its direction of motion is clearly perpendicular to the plane PON, and since the perpendicular distance of N from OP is $ON \cos\gamma$, the velocity of N is $\omega . ON \cos\gamma$. Resolving this along Ox we have $-\omega y \cos\gamma$. But by Prop. III. the velocity parallel to the axis of x is $-\omega_3 y$;

$$\therefore \ \omega_3 = \omega \cos\gamma.$$

By similar reasoning we can prove

$$\omega_1 = \omega \cos\alpha,$$

$$\omega_2 = \omega \cos\beta;$$

adding the squares of these three equations we get

$$\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2.$$

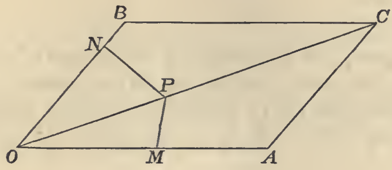
94. Prop. V. *If two angular velocities about two axes OA, OB, be represented in magnitude and direction by the two lengths OA, OB; then the diagonal OC of the parallelogram constructed on OA, OB, as sides, will be the resultant axis of rotation, and its length will represent the magnitude of the resultant angular velocity.* This Prop. is usually called "The Parallelogram of Angular Velocities."

Let P be any point in OC, and let PM, PN be drawn perpendicular to OA, OB. Then the velocity of any point P is perpendicular to the plane AOB, and is represented by

$$\begin{aligned} &OA . PM - OB . PN \\ &= OP . \{OA . \sin COA - OB . \sin COB\} \\ &= 0. \end{aligned}$$

Therefore the point P is at rest and OC is the resultant axis of rotation.

Let ω be the angular velocity about OC, then the velocity



of any point A in OA is perpendicular to the plane AOB and equal to $\omega . OA . \sin COA$. But it is also equal to

$$OB . OA \sin BOA;$$

$$\therefore \omega = OB . \frac{\sin BOA}{\sin COA}$$

$$= OC.$$

Hence the angular velocity about OC is represented in magnitude by OC.

95. Prop. VI. *Every motion of a rigid body may be represented by a combination of the two following motions.*

First. A motion of translation whereby every particle is moved parallel to the direction of motion of any assumed point rigidly connected with the body, and with the same velocity.

Secondly. A motion of rotation of the whole body about some axis through this assumed point.

This may be proved as follows. It is evident that the change of position of the body can be effected by *first* moving any point O from its old to its new position by a motion of translation, and *secondly*, retaining this point O fixed, by moving any two points of the body not in one straight line with O, into their new positions. This last motion has been proved in Prop. I. to be equivalent to rotation about a single axis through O.

Let ω_1, ω_2, ω_3, be the angular velocities of the body about three rectangular axes meeting in any assumed point O, u, v, w the linear velocities of O parallel to these axes. Then the velocities of any point whose co-ordinates are x, y, z, are

$$\left.\begin{aligned}\frac{dx}{dt} &= u + \omega_2 z - \omega_3 y\\ \frac{dy}{dt} &= v + \omega_3 x - \omega_1 z\\ \frac{dz}{dt} &= w + \omega_1 y - \omega_2 x\end{aligned}\right\}.$$

96. Since we may begin by assuming a great number of different points as origin, the motion of a body from one position to another may be represented in a great many different ways. It remains to explain the connection that exists between these different representations. The analogy which exists between forces and rotations will enable us to do this. We have seen that forces and rotations have the same law of composition and resolution, and therefore every proposition concerning forces has its corresponding proposition in the theory of rotations. But in order to make any use of these results it will be necessary to determine the analogue of a couple. The following proposition will show that it corresponds to a motion of *translation*.

97. PROP. VII. *A body has coexistent angular velocities ω and ω' about two parallel axes OA, $O'A'$ distant h from each other, to find the resulting motion.*

Let OA be taken as the axis of x, and the plane of OA, $A'O'$ as plane of xy. Let P be any point whose co-ordinates are x, y, z. By Prop. III. the velocities of P resolved parallel to the axes are respectively

$$\frac{dx}{dt}=0, \quad \frac{dy}{dt}=-(\omega+\omega')\,z, \quad \frac{dz}{dt}=\omega y+\omega'(y-h).$$

If these three quantities be made to vanish simultaneously, we must have

$$z=0, \quad y=\frac{h\omega'}{\omega+\omega'}.$$

The motion is therefore one of rotation about a single axis, and the position of the axis is determined by the above values of y and z. If Ω be the resultant angular velocity about it, we have $\frac{dy}{dt}=-\Omega z$, and

$$\therefore\ \Omega=\omega+\omega'.$$

It appears therefore that the resultant axis of rotation may be found by exactly the same process as that by which we determine the resultant of two *forces* ω, ω' acting along OA, $O'A'$.

If $\omega=-\omega'$ the resultant vanishes, but y becomes at the same time infinite. The motion in this case is given by

$$\frac{dx}{dt}=0, \quad \frac{dy}{dt}=0, \quad \frac{dz}{dt}=h\omega,$$

that is, the motion is one of translation parallel to the axis of z.

98. We may deduce as a corollary to this proposition, that a motion of rotation ω about an axis OA is equivalent to a motion of rotation ω about a parallel axis $O'A'$ plus a motion of translation $h\omega$ perpendicular to the plane OA, $O'A'$, and in the direction in which O' moves.

99. It is proved in Statics that a system of forces is generally equivalent to a single force and a single couple, and that these may be reduced to a resultant R, acting along a line called the central axis, and a couple G about that axis.

Or they may also be reduced to a resultant R of the same magnitude as before, acting along any line parallel to the central axis at any chosen distance c from it, together with a couple G' about an axis perpendicular to the line whose length is c, and inclined to the resultant R at an angle θ. Then we know that $G' = \sqrt{G^2 + R^2c^2}$, and is a minimum when $c = 0$, and also that $\tan\theta = \frac{Rc}{G}$.

The same train of reasoning by which these results were established, will establish the following proposition. The general motion of a body having been reduced to a motion of translation and one of rotation, these are equivalent to a motion of rotation ω about a line called the central axis, and a translation along that axis. Or they may also be reduced to a rotation ω of the same magnitude ω about any line parallel to the central axis, and at any chosen distance c from it, together with a translation V' along a line perpendicular to the line c, and inclined to the axis of ω at an angle θ. Then we know that $V' = \sqrt{V^2 + c^2\omega^2}$ and is a minimum when $c = 0$, and also that $\tan\theta = \frac{c\omega}{V}$.

In a similar manner many other propositions may be established.

100. Prop. VIII. *The motion of a body being reduced to a motion of translation and rotation, it is required to find the condition that the motion may be one of rotation only about some axis, and to find that axis.*

This evidently corresponds to the proposition in Statics, "To find the condition that a system of forces may be reduced to a single resultant," and the required condition may be inferred from the result there obtained.

But we may also reason thus. Let ω_1, ω_2, ω_3 be the angular velocities about the three axes of co-ordinates; u, v, w the linear velocities of the origin. Then the velocities of any point P on the single axis of rotation must vanish. Hence

$$\left.\begin{aligned} \frac{dx}{dt} &= u + \omega_2 z - \omega_3 y = 0 \\ \frac{dy}{dt} &= v + \omega_3 x - \omega_1 z = 0 \\ \frac{dz}{dt} &= w + \omega_1 y - \omega_2 x = 0 \end{aligned}\right\} \dots\dots\dots\dots\dots (1).$$

If the motion be one of rotation only, these are the equations to that axis, and they must therefore be really equivalent to only two. Multiplying these respectively by ω_1, ω_2, ω_3, we get

$$u\omega_1 + v\omega_2 + w\omega_3 = 0.$$

This is therefore a *necessary* condition, but it is not *sufficient*, for it is evident that the equations (1) cannot be satisfied if *all* the three quantities ω_1, ω_2, ω_3 be zero.

SECT. II. *The motion of a body of given form under any forces.*

101. PROP. I. *To determine the general equations of motion of a body about a fixed point.*

Let the fixed point O be taken as origin, and let x, y, z be the co-ordinates at time t of any particle m referred to any rectangular axes fixed in space. Let Xm, Ym, Zm be the impressed forces acting on this element, and let L, M, N be the moments of all these forces about the axes of co-ordinates, and let P, Q, R be the pressures of the fixed point on the body.

Then by D'Alembert's Principle, if the effective forces $m\dfrac{d^2x}{dt^2}$, $m\dfrac{d^2y}{dt^2}$, $m\dfrac{d^2z}{dt^2}$ be applied to every particle m in a reversed direction, there will be equilibrium between these forces and the impressed forces. Taking moments therefore about the axes, we have

$$\left.\begin{aligned}\Sigma m\left(y\frac{d^2z}{dt^2}-z\frac{d^2y}{dt^2}\right)&=L\\ \Sigma m\left(z\frac{d^2x}{dt^2}-x\frac{d^2z}{dt^2}\right)&=M\\ \Sigma m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right)&=N\end{aligned}\right\}\text{..............}\quad\text{(I)}.$$

Resolving parallel to the axes, we have

$$\left.\begin{aligned}\Sigma m\frac{d^2x}{dt^2}&=P+\Sigma mX\\ \Sigma m\frac{d^2y}{dt^2}&=Q+\Sigma mY\\ \Sigma m\frac{d^2z}{dt^2}&=R+\Sigma mZ\end{aligned}\right\}\text{................}\quad\text{(II)}.$$

To simplify these equations, let ω_x, ω_y, ω_z be the angular velocities about the axes. Then

$$\frac{dx}{dt}=\omega_y z-\omega_z y,$$

$$\frac{dy}{dt}=\omega_z x-\omega_x z,$$

$$\frac{dz}{dt}=\omega_x y-\omega_y x;$$

$$\therefore\ \frac{d^2x}{dt^2}=z\frac{d\omega_y}{dt}-y\frac{d\omega_z}{dt}+\omega_y(\omega_x y-\omega_y x)-\omega_z(\omega_z x-\omega_x z),$$

$$\frac{d^2y}{dt^2}=x\frac{d\omega_z}{dt}-z\frac{d\omega_x}{dt}+\omega_z(\omega_y z-\omega_z y)-\omega_x(\omega_x y-\omega_y x).$$

Substituting in the last of equations (I) we get

$$\left.\begin{aligned}\Sigma m\,(x^2+y^2)\,\frac{d\omega_z}{dt}-\Sigma myz\,.\,\frac{d\omega_y}{dt}-\Sigma mxz\,.\,\frac{d\omega_x}{dt}\qquad\qquad\qquad\qquad\\ -\Sigma mxy\,.\,(\omega_x{}^2-\omega_y{}^2)+\Sigma m\,(x^2-y^2)\,\omega_x\omega_y-\Sigma myz\,.\,\omega_x\omega_z\\ +\Sigma mxz\,.\,\omega_y\omega_z\end{aligned}\right\}=N.$$

The other two equations may be treated in the same manner.

The original equations (I) cannot be used because they contain an infinite number of unknown accelerations. By this transformation they have all been reduced to depend on three unknown quantities, viz. ω_x, ω_y, ω_z. But the equations thus obtained are so complicated as to be practically useless. To simplify them still further take axes OA, OB, OC, fixed in the body, and coincident with the principal axes of the body at the point O, and let ω_1, ω_2, ω_3 be the angular velocities about these axes.

Since the axes Ox, Oy, Oz are perfectly arbitrary, take them so that the axes OA, OB, OC are passing through them at the moment under consideration. Then $\omega_x=\omega_1$, $\omega_y=\omega_2$, $\omega_z=\omega_3$, and the last equation reduces to

$$\Sigma m\,(x^2+y^2)\,\frac{d\omega_z}{dt}+\Sigma m\,(x^2-y^2)\,\omega_1\omega_2=N.$$

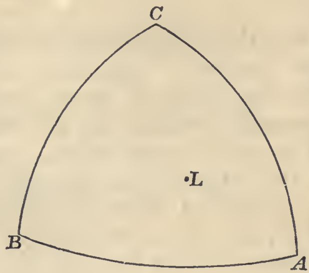
102. We have now to find the relation between $\frac{d\omega_z}{dt}$ and $\frac{d\omega_3}{dt}$*. Let A, B, C, be the points in which the principal axes cut a sphere whose centre is at the fixed point. Let OL be any other axis, and let Ω be the angular velocity about it. Let the angles LOA, LOB, LOC be called respectively α, β, γ. Then

$$\Omega=\omega_1\cos\alpha+\omega_2\cos\beta+\omega_3\cos\gamma\,;$$

* This demonstration of the equality of $\frac{d\omega_z}{dt}$ and $\frac{d\omega_3}{dt}$ is due to Professor Slesser, of Queen's College, Belfast.

$$\therefore \frac{d\Omega}{dt} = \frac{d\omega_1}{dt}\cos\alpha + \frac{d\omega_2}{dt}\cos\beta + \frac{d\omega_3}{dt}\cos\gamma$$
$$- \omega_1 \sin\alpha \frac{d\alpha}{dt} - \omega_2 \sin\beta \frac{d\beta}{dt} - \omega_3 \sin\gamma \frac{d\gamma}{dt}.$$

Now let the line OL be fixed in space and coincide with OC



at the moment under consideration. Then $\alpha = \frac{\pi}{2}$, $\beta = \frac{\pi}{2}$, $\gamma = 0$; therefore

$$\frac{d\Omega}{dt} = \frac{d\omega_3}{dt} - \omega_1 \frac{d\alpha}{dt} - \omega_2 \frac{d\beta}{dt}.$$

Also $\frac{d\alpha}{dt}$ is the rate at which A separates from a *fixed* point at C, which is clearly ω_2. Similarly $\frac{d\beta}{dt} = -\omega_1$. Hence

$$\frac{d\Omega}{dt} = \frac{d\omega_3}{dt}.$$

Thus $$\frac{d\omega_x}{dt} = \frac{d\omega_1}{dt},\quad \frac{d\omega_y}{dt} = \frac{d\omega_2}{dt},\quad \frac{d\omega_z}{dt} = \frac{d\omega_3}{dt}.$$

The equation $\frac{d\omega_z}{dt} = \frac{d\omega_3}{dt}$ may appear at first sight to be a mere truism, but it is not so; ω_3 denotes the angular velocity of the body about the axis OC *fixed in the body*, ω_z denotes the angular velocity about a line Oz *fixed in space*, and determined by the condition that *at the time t*, OC coincides with it. At the time $t+\delta t$, OC will have separated from Oz, and we cannot therefore assert *a priori* that the angular velocity about OC will continue to be the same as that about Oz. The above investigation proves that this is the case as far as the first order of small quantities.

Substituting for $\frac{d\omega_z}{dt}$ the equation of motion becomes

$$\Sigma m\,(x^2+y^2)\,\frac{d\omega_3}{dt}+\Sigma m\,(x^2-y^2)\,\omega_1\omega_2=N.$$

Let A, B, C be the moments of inertia of the body about the principal axes at O; then the three equations of motion are

$$\left.\begin{aligned} A\frac{d\omega_1}{dt}-(B-C)\,\omega_2\omega_3&=L\\ B\frac{d\omega_2}{dt}-(C-A)\,\omega_3\omega_1&=M\\ C\frac{d\omega_3}{dt}-(A-B)\,\omega_1\omega_2&=N\end{aligned}\right\}\text{............ (III).}$$

These are called Euler's Equations.

103. *To determine the geometrical equations connecting the motion of the body in space with the angular velocities of the body about the three moving axes, OA, OB, OC.*

Let the fixed point be taken as the centre of a sphere of rad. unity; let X, Y, Z, A, B, C be the points in which the sphere is cut by the fixed and moving axes respectively. Let ZC, BA produced if necessary, meet in E. Let the angle $XZC=\psi$, $ZC=\theta$, $ECA=\phi$. It is required to determine the geometrical relations between θ, ϕ, ψ, and ω_1, ω_2, ω_3.

It is evident that the velocities of the point of the body coinciding with C are

$$\frac{d\psi}{dt}\sin\theta \text{ perpendicular to } ZC,$$

$$\text{and } \frac{d\theta}{dt} \text{ along } ZC,$$

and the velocity along EA of the point of the body coinciding with A is

$$\frac{d\psi}{dt}\sin ZE \text{ (due to variation of } \psi)$$

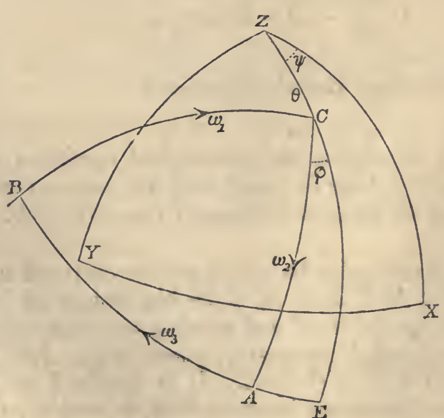
$$+\frac{d\phi}{dt}\sin CE \text{ (due to variation of } \phi),$$

or since $CE=\frac{\pi}{2}$,

$$\frac{d\psi}{dt}\cos\theta+\frac{d\phi}{dt}.$$

Now the velocities of the point of the body coinciding with C, are also

$$\omega_2 \text{ along } CA,$$
$$\omega_1 \text{ along } BC,$$



and the velocity along EA of the point coinciding with A is ω_3; hence resolving the latter set so as to make them coincide with the former,

$$\left.\begin{aligned} \frac{d\psi}{dt}\sin\theta &= -\omega_1\cos\phi + \omega_2\sin\phi \\ \frac{d\theta}{dt} &= \omega_1\sin\phi + \omega_2\cos\phi \\ \frac{d\psi}{dt}\cos\theta + \frac{d\phi}{dt} &= \omega_3 \end{aligned}\right\}\text{......(IV).}$$

The dynamical equations (III) and the geometrical equations (IV) are sufficient for the determination of the whole motion.

104. *To determine the pressure on the fixed point.*

If $\bar{x}, \bar{y}, \bar{z}$, be the co-ordinates of the centre of gravity, the equations (II) reduce to the form

$$M\,.\frac{d^2\bar{x}}{dt^2} = P + \Sigma\,(mX),$$

and two similar equations. It is necessary to express $\frac{d^2\bar{x}}{dt^2}$ the acceleration along a fixed straight line in terms of ω_x, ω_y, ω_z. This has been already done, and we have

$$M\left\{\bar{z}\,\frac{d\omega_y}{dt} - \bar{y}\,\frac{d\omega_z}{dt} + \omega_y(\omega_x\bar{y} - \omega_y\bar{x}) - \omega_z(\omega_z\bar{x} - \omega_x\bar{z})\right\} = P + \Sigma(mX),$$

and two similar equations.

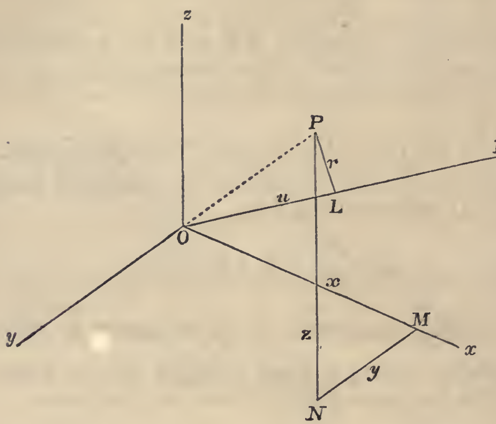
105. It appears from Euler's Equations that the whole changes of $\omega_1, \omega_2, \omega_3$ are not due merely to the direct action of the forces, but in part are due to the centrifugal force of the particles tending to carry them away from the axis about which they are revolving. For consider the equation

$$\frac{d\omega_3}{dt} = \frac{N}{C} + \frac{A-B}{C}\,\omega_1\omega_2.$$

Of the increase $d\omega_3$ in the time dt, the part $\frac{N}{C}dt$ is due to the direct action of the forces whose moment is N, and the part $\frac{A-B}{C}\,\omega_1\omega_2\,dt$ is due to the centrifugal force. This may be proved as follows.

If a body be rotating about an axis OI with an angular velocity ω, then the moment of the centrifugal forces of the whole body about the axis Oz is $(A-B)\,\omega_1\omega_2$.

Let P be the position of any particle m and let x, y, z



be its co-ordinates. Let PL be a perpendicular on OI, let $OL=u$, and $PL=r$. Then the centrifugal force of the particle m is $\omega^2 rm$ tending from OI.

The force $\omega^2 rm$ is evidently equivalent to the *four* forces $\omega^2 xm$, $\omega^2 ym$, $\omega^2 zm$, and $-\omega^2 um$ acting at P parallel to x, y, z, and u respectively.

$$\left.\begin{array}{l}\text{The moment of } \omega^2 xm \text{ round } Oz = -\omega^2 xym \\ \ldots\ldots\ldots\ldots\ldots\ldots\ \omega^2 ym\ \ldots\ldots\ldots\ldots = \omega^2 xym \\ \ldots\ldots\ldots\ldots\ldots\ldots\ \omega^2 zm\ \ldots\ldots\ldots\ldots = 0\end{array}\right\};$$

these three therefore produce no effect.

The force $-\omega^2 um$ parallel to OI is equivalent to the three, $-\omega\omega_1 um$, $-\omega\omega_2 um$, $-\omega\omega_3 um$, acting at P parallel to the axes, and their moment round Oz is evidently

$$\omega um(\omega_1 y - \omega_2 x).$$

Now since the direction-cosines of OI are $\frac{\omega_1}{\omega}$, $\frac{\omega_2}{\omega}$, $\frac{\omega_3}{\omega}$, therefore by projecting the broken line x, y, z on OI, we get

$$u = \frac{\omega_1}{\omega}.x + \frac{\omega_2}{\omega}y + \frac{\omega_3}{\omega}z;$$

therefore substituting for u, the moment of centrifugal forces about Oz is

$$= (\omega_1 y - \omega_2 x)(\omega_1 x + \omega_2 y + \omega_3 z)\,m,$$

$$= \omega_1^2 xy + \omega_1\omega_2 y^2 + \omega_1\omega_3 yz - \omega_1\omega_2 x^2 - \omega_2^2 xy - \omega_2\omega_3 xz\,.\,m.$$

Writing Σ before every term, and supposing the axes of x, y, z, to be principal axes, then the moment of the centrifugal forces about the principal axis Oz

$$= \omega_1\omega_2 \Sigma m(y^2 - x^2)$$

$$= \omega_1\omega_2 (A - B).$$

106. The equations of Euler determine the motion with reference to axes fixed in the body. The motions of these axes being unknown, the moments about them must be found without any limitation as to their position. Hence the quantities L, M, N will generally be very complicated functions of θ, ϕ, ψ. When Euler's equations are joined to the geometrical equations (IV), the eliminations to be performed are then so complicated as to be practically impossible. It becomes necessary therefore to inquire whether the two sets of equations can be simplified by referring the motion to axes moving in the body. This simplification can generally be effected when either two or all three of the principal moments of inertia at the fixed point are equal.

107. Prop. II. *To discuss the different forms which Euler's general equations of motion assume when two of the principal moments of inertia at the fixed point are equal to each other.*

Suppose $A = B$. Then instead of choosing fixed axes Ox and Oy, we may choose axes $O\xi$ and $O\eta$ which move in any manner round the third axis OC which remains fixed in the body.

Let χ be the angle the axis of ξ makes with x, and let ω_ξ, ω_η be the angular velocities about the axes ξ, η. Then

$$\omega_x = \omega_\xi \cos\chi - \omega_\eta \sin\chi,$$

$$\therefore \frac{d\omega_x}{dt} = \frac{d\omega_\xi}{dt}\cos\chi - \frac{d\omega_\eta}{dt}\sin\chi - \omega_\xi \sin\chi\frac{d\chi}{dt} - \omega_\eta\cos\chi\frac{d\chi}{dt}.$$

Let the axis of x be taken so that the axis of ξ is passing through it at the moment under consideration, then $\chi = 0$,

$$\therefore \frac{d\omega_x}{dt} = \frac{d\omega_\xi}{dt} - \omega_\eta\frac{d\chi}{dt}.$$

Similarly by putting $\chi = -\frac{\pi}{2}$ we get

$$\frac{d\omega_y}{dt} = \frac{d\omega_\eta}{dt} + \omega_\xi\frac{d\chi}{dt}.$$

The axes of $O\xi$, $O\eta$ have been supposed to be moving from Ox to Oy, and that expression contains the negative sign which treats of the axis of x.

Euler's equations now become

$$\left.\begin{aligned} A\left(\frac{d\omega_1}{dt} - \omega_2\frac{d\chi}{dt}\right) - (A - C)\,\omega_2\omega_3 &= L \\ A\left(\frac{d\omega_2}{dt} + \omega_1\frac{d\chi}{dt}\right) + (A - C)\,\omega_1\omega_3 &= M \\ C\frac{d\omega_3}{dt} &= N \end{aligned}\right\} \ldots\ldots (\text{V}).$$

Two of the geometrical equations are the same as the first two, given in IV. Since the axes are moving in the body with an angular velocity $\frac{d\chi}{dt}$, in the third equation we must put $\frac{d\phi}{dt} - \frac{d\chi}{dt}$ for $\frac{d\phi}{dt}$. Hence the three equations are

$$\left.\begin{aligned} \frac{d\psi}{dt}\sin\theta &= -\omega_1\cos\phi + \omega_2\sin\phi \\ \frac{d\theta}{dt} &= \omega_1\sin\phi + \omega_2\cos\phi \\ \frac{d\psi}{dt}\cos\theta + \frac{d\phi}{dt} &= \omega_3 + \frac{d\chi}{dt} \end{aligned}\right\} \ldots \text{(VI.)}$$

108. There are two cases in which these equations become much simpler.

First. Since $\frac{d\chi}{dt}$ is perfectly arbitrary, let it be chosen $= -\omega_3$. Then the above equations reduce to

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} + C\omega_2\omega_3 &= L \\ A\frac{d\omega_2}{dt} - C\omega_3\omega_1 &= M \\ C\frac{d\omega_3}{dt} &= N \end{aligned}\right\}.$$

The third geometrical equation takes the form

$$\frac{d\psi}{dt}\cos\theta + \frac{d\phi}{dt} = 0.$$

Secondly. Let $\frac{d\chi}{dt}$ be so chosen that the axes Oz, OC, and OA shall be in one plane, then $\phi = 0$, and the geometrical equations become

$$\left.\begin{aligned} \frac{d\theta}{dt} &= \omega_2 \\ \frac{d\psi}{dt}\sin\theta &= -\omega_1 \\ -\frac{d\chi}{dt} + \frac{d\psi}{dt}\cos\theta &= \omega_3 \end{aligned}\right\}.$$

The dynamical equations are the same as (V).

If $L = 0$ and M be any function of θ these equations admit of complete solution, as shown in the following example.

109. PROB. *A body, two of whose principal moments at the centre of gravity are equal, moves about some fixed point O in the axis of unequal moment, under the action of gravity. Determine the motion.*

This is the problem of a top spinning on a perfectly rough horizontal plane. In the investigation a top is sometimes spoken of, for convenience of reference, but the process is quite general.

Let the axis of Oz be vertical. Let the axis of unequal moment at the centre of gravity be the axis OC, and let this be called the axis of the body. Let h be the distance of the centre of gravity of the body from the fixed point O, and let the mass of the body be taken as unity. Then by the second part of Art. 108, the equations of motion are

$$\left.\begin{aligned} A\left(\frac{d\omega_1}{dt} - \omega_2\frac{d\chi}{dt}\right) - (A - C)\,\omega_2\omega_3 &= 0 \\ A\left(\frac{d\omega_2}{dt} + \omega_1\frac{d\chi}{dt}\right) + (A - C)\,\omega_3\omega_1 &= gh\sin\theta \\ C\frac{d\omega_3}{dt} &= 0 \end{aligned}\right\} \quad \text{.......(1)}.$$

$$\left.\begin{aligned} \frac{d\theta}{dt} &= \omega_2 \\ \frac{d\psi}{dt}\sin\theta &= -\omega_1 \\ -\frac{d\chi}{dt} + \frac{d\psi}{dt}\cos\theta &= \omega_3 \end{aligned}\right\} \text{.....................(2).}$$

Eliminating $\frac{d\chi}{dt}$ and $\frac{d\psi}{dt}$ we get

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} + A\cot\theta\omega_1\omega_2 + C\omega_2\omega_3 &= 0 \\ A\frac{d\omega_2}{dt} - A\cot\theta\omega_1^2 - C\omega_1\omega_3 &= gh.\sin\theta \end{aligned}\right\} \text{...........(3),}$$

and $\omega_3 = n$ is a constant quantity.

Putting $\omega_2 = \frac{d\theta}{dt}$, the first of these equations becomes

$$A\left(\frac{d\omega_1}{dt}\sin\theta + \cos\theta\frac{d\theta}{dt}\omega_1\right) + Cn\sin\theta\frac{d\theta}{dt} = 0.$$

Integrating we get

$$A\omega_1\sin\theta - Cn\cos\theta = \alpha \text{..................(4),}$$

where α is an arbitrary constant.

Multiplying the first of equations (3) by ω_1 and the second by ω_2, and adding, we get

$$A\left(\omega_1\frac{d\omega_1}{dt} + \omega_2\frac{d\omega_2}{dt}\right) = gh\sin\theta\frac{d\theta}{dt};$$

$$\therefore\ A(\omega_1^2 + \omega_2^2) = -2gh.\cos\theta + \beta \text{ ....(5),}$$

where β is another arbitrary constant. These two equations (4) and (5) might also have been deduced from the principles of Conservation of Areas and Vis Viva.

It is evident that having now found ω_2 in terms of θ, we can find θ in terms of t by means of equation (2). The integration will be found to be of the form

$$t=\int\frac{d\cos\theta}{\sqrt{\alpha+\beta\cos\theta+\gamma\cos^2\theta+\delta\cos^3\theta}}.$$

Also $\frac{d\psi}{dt}$ the rate at which the top goes round the vertical can be found from the equation

$$\frac{d\psi}{dt}=-\frac{\omega_1}{\sin\theta}=-\frac{\alpha+Cn\cos\theta}{A\sin^2\theta},$$

showing that when the body is nearest the vertical, it goes round the vertical with the greatest angular velocity.

The equation to determine the motion of the axis of the body is

$$\frac{(\alpha+Cn\cos\theta)^2}{A^2\sin^2\theta}+\left(\frac{d\theta}{dt}\right)^2=\frac{\beta-2gh\cos\theta}{A} \quad\text{..........(6).}$$

From this equation we see that θ can never vanish unless $\alpha=-Cn$, for the left-hand side of the above equation would then become infinite. Hence the axis of the body cannot, in general, become vertical. Suppose the body to be set in motion in any way with its axis at an inclination i to the vertical. The axis will begin to approach or to fall away from the vertical according as the initial value of $\frac{d\theta}{dt}$ or ω_2 is negative or positive. This motion will continue until $\frac{d\theta}{dt}$ vanishes: it is evident from the equation that the axis will then begin to return, and will oscillate between two limiting angles. To find these limits we have the equation

$$(\alpha+Cn\cos\theta)^2-A(1-\cos^2\theta)(\beta-2gh\cos\theta)=0\text{....(7).}$$

This is a cubic equation to determine $\cos\theta$. It will be necessary to examine its roots. When $\cos\theta=-1$, the left-hand side is *positive;* when $\cos\theta = \cos i$, since the initial value of $\left(\frac{d\theta}{dt}\right)^2$ is essentially positive, the left-hand side is either zero or *negative:* hence the equation has one real root between $\cos\theta = -1$ and $\cos\theta = \cos i$. Again, the left-hand side is *positive* when $\cos\theta = +1$, and is *negative* when $\cos\theta = \infty$. Hence there is another real root between $\cos\theta = \cos i$ and $\cos\theta = 1$, and a third root greater than unity. This last root is inadmissible.

If the initial values of ω_1, ω_2 are zero, we have by (4) and (5) $\alpha = -Cn\cos i$, $\beta = 2gh\cos i$. Hence the equation becomes

$$(\cos i - \cos\theta)^2 = \frac{2ghA}{C^2n^2} \cdot (1 - \cos^2\theta)(\cos i - \cos\theta).$$

Putting $\frac{C^2n^2}{2ghA} = 2p$, the roots of this equation are

$$\left.\begin{aligned} \cos\theta &= \cos i \\ \cos\theta &= p - \sqrt{1 - 2p\cos i + p^2} \end{aligned}\right\} \dots\dots\dots\dots\dots (8).$$

The value $\cos\theta = p + \sqrt{1 - 2p\cos i + p^2}$ is always greater than unity, for it is clearly decreased by putting unity for $\cos i$, and its value is then $p + 1 - p = 1$. The body will therefore oscillate between the values of θ given by the equations (8).

If a top be set spinning on a *perfectly smooth horizontal plane*, its motion may be determined in the same way. The equations (2) and the left-hand sides of the equations (1) will be the same as before, and the whole process will be very similar.

110. If a top be spun on the ground it is seen to raise itself up into a vertical position, and to remain so for some little time. But equation (5) shows that $h\cos\theta$, the height of the centre of gravity, can never exceed a certain quantity. If the initial values of $\omega_1\omega_2$ are small, this quantity may be considerably less than h, and in this case the top can never become nearly vertical. If the ground were smooth, an equation similar to (5) can be easily proved to exist, and the same conclusion

will follow. But the case of a real top differs from the problem we have been considering in two particulars. The apex is not a mathematical point, and the friction is not generally sufficient to prevent sliding. The apex of the top both rolls and slides on the ground.

111. PROP. III. *To determine in what cases we can take moments about the instantaneous axis as if it were a fixed axis.*

Following the usual notation the equations of motion are

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} - (B-C)\,\omega_2\omega_3 &= L \\ B\frac{d\omega_2}{dt} - (C-A)\,\omega_3\omega_1 &= M \\ C\frac{d\omega_3}{dt} - (A-B)\,\omega_1\omega_2 &= N \end{aligned}\right\}.$$

Let l, m, n be the direction cosines of the instantaneous axis, I and Ω the moment of inertia and angular velocity about it; then $\omega_1 = l\Omega$, $\omega_2 = m\Omega$, $\omega_3 = n\Omega$. Substituting in the above equations, we get

$$\left.\begin{aligned} Al^2\frac{d\Omega}{dt} - (B-C)\,lmn\,\Omega^2 &= lL - Al\frac{dl}{dt}\Omega \\ Bm^2\frac{d\Omega}{dt} - (C-A)\,lmn\,\Omega^2 &= mM - Bm\frac{dm}{dt}\Omega \\ Cn^2\frac{d\Omega}{dt} - (A-B)\,lmn\,\Omega^2 &= nN - Cn\frac{dn}{dt}\Omega \end{aligned}\right\};$$

adding these,

$$I\frac{d\Omega}{dt} = P - \Omega\left(Al\frac{dl}{dt} + Bm\frac{dm}{dt} + Cn\frac{dn}{dt}\right),$$

where P is the moment of all the forces about the instantaneous axis. Hence the equation

$$I\frac{d\Omega}{dt} = P$$

will hold whenever

$$Al\frac{dl}{dt} + Bm\frac{dm}{dt} + Cn\frac{dn}{dt} = 0.$$

This is satisfied in the three following cases:

First. When l, m, n, are constants.

Secondly. When $A = B$ and n is constant, i.e. when the instantaneous axis always makes the same angle with the axis of unequal moment of inertia.

Thirdly. When $A = B = C$; as for example, if the body be a cube or a sphere, and the fixed point be the centre of gravity.

112. Ex. *A right cone is placed with its slant side on a perfectly rough inclined plane, and rolls on it under the action of gravity. It is required to find the motion.*

Let C be the axis of unequal moment of inertia, then since the cone rolls, the instantaneous axis makes a constant angle with the axis C. Hence we can take moments about the instantaneous axis.

Let the fixed vertex of the cone be the origin, and let ϕ be the angle the side of the cone in contact with the plane makes with the direction in which gravity acts when resolved along the plane. Let 2α be the angle, h the height of the cone, β the inclination of the plane to the horizon. Then

$$I\frac{d\Omega}{dt} = -Mg\sin\beta \cdot \frac{3}{4}h\sin\alpha \cdot \sin\phi,$$

$$\text{also } \Omega\sin\alpha = \frac{d\phi}{dt} \cdot \cos\alpha;$$

$$\therefore \frac{d^2\phi}{dt^2} = -\frac{3}{4}\frac{Mgh\sin^2\alpha\sin\beta}{I\cos\alpha} \cdot \sin\phi.$$

This equation can be easily integrated and the whole motion found. If the cone just make complete revolutions, the angular velocity at the lowest point will be given by the equation

$$\Omega^2 = \frac{20g \sin\beta}{h} \cdot \frac{\cos\alpha}{\sin^2\alpha + 6\cos^2\alpha}.$$

In the same way we may find the motion of a cone rolling under the action of any forces on another perfectly rough cone; or, its vertex being fixed, rolling on any rough curve. These motions may also be found by means of the principle of vis viva.

113. Prop. IV. *To discuss the different forms which Euler's general equations of motion assume when the three principal moments of inertia at the fixed point are equal to each other.*

There are three sets of axes such that when the motion is referred to them, the equations take a simple form.

First. We may choose axes fixed in space. For since every axis is a principal axis in the body the general equations in Art. 101 take the simple form

$$\left.\begin{aligned} \frac{d\omega_x}{dt} &= \frac{L}{A} \\ \frac{d\omega_y}{dt} &= \frac{M}{A} \\ \frac{d\omega_z}{dt} &= \frac{N}{A} \end{aligned}\right\},$$

and the geometrical equations (IV) are no longer wanted.

Secondly. We may choose one axis as that of C fixed in space, and make the other two moveround it in any manner, as shown in Art. 107. The equations of motion then become

$$\left.\begin{aligned} \frac{d\omega_x}{dt} - \omega_y \frac{d\chi}{dt} &= \frac{L}{A} \\ \frac{d\omega_y}{dt} + \omega_x \frac{d\chi}{dt} &= \frac{M}{A} \\ \frac{d\omega_z}{dt} &= \frac{N}{A} \end{aligned}\right\}.$$

Thirdly. We can take as axes any three straight lines at right angles moving in space in any proposed manner. To effect this the two following propositions will be necessary.

114. Prop. A. *To determine the equations of motion of a particle with reference to axes moving in any manner about a fixed origin.*

Let the moving axes be Ox, Oy, Oz, and let their motion be given by the angular velocities θ_1, θ_2, θ_3, about the axes Ox, Oy, Oz, respectively. Let OL be any line *fixed in space* making with Ox, Oy, Oz, the angles α, β, γ. Let u, v, w be the velocities of any point P along the axes, and let V be the velocity resolved along OL. Then

$$V = u\cos\alpha + v\cos\beta + w\cos\gamma;$$

$$\therefore \frac{dV}{dt} = \frac{du}{dt}\cos\alpha + \frac{dv}{dt}\cos\beta + \frac{dw}{dt}\cos\gamma$$
$$- u\sin\alpha\frac{d\alpha}{dt} - v\sin\beta\frac{d\beta}{dt} - w\sin\gamma\frac{d\gamma}{dt}.$$

Now because OL is fixed in space, $\frac{dV}{dt}$ is the accelerating effect of the force along OL.

Let X, Y, Z be the accelerating effects of the impressed forces along the axes. Then taking OL, so that the axis of z is passing through it at the moment under consideration, we have $\alpha = \frac{\pi}{2}$, $\beta = \frac{\pi}{2}$, $\gamma = 0$,

$$\therefore Z = \frac{dw}{dt} - u\frac{d\alpha}{dt} - v\frac{d\beta}{dt}.$$

But $\frac{d\alpha}{dt}$ is the rate at which OA separates from a fixed axis OL at Oz, and this is clearly θ_2;

$$\therefore \frac{d\alpha}{dt} = \theta_2 \text{ and } \frac{d\beta}{dt} = -\theta_1;$$

$$\therefore\ Z = \frac{dw}{dt} - u\theta_2 + v\theta_1.$$

Similarly,

$$X = \frac{du}{dt} - v\theta_3 + w\theta_2,$$

$$Y = \frac{dv}{dt} - w\theta_1 + u\theta_3.$$

Cor. In the same way it may be shown that the velocities parallel to the moving axes are given by

$$u = \frac{dx}{dt} - y\theta_3 + z\theta_2,$$

$$v = \frac{dy}{dt} - z\theta_1 + x\theta_3,$$

$$w = \frac{dz}{dt} - x\theta_2 + y\theta_1.$$

115. Prop. B. *To determine the equations of rotation with reference to axes moving in any manner**.

The preceding proposition is a simple corollary from the parallelogram of velocities. The result will therefore be true for any other kind of magnitude which also obeys the "parallelogram law." In fact the demonstration is exactly the same. Now angular velocities do obey this law. Hence the following equations are clearly true:

$$\frac{L}{A} = \frac{d\omega_1}{dt} - \omega_2\theta_3 + \omega_3\theta_2,$$

$$\frac{M}{A} = \frac{d\omega_2}{dt} - \omega_3\theta_1 + \omega_1\theta_3,$$

$$\frac{N}{A} = \frac{d\omega_3}{dt} - \omega_1\theta_2 + \omega_2\theta_1.$$

* The results of these two propositions were first published by Mr Hayward, in Vol. x. of the *Camb. Phil. Trans.* The latter set were subsequently independently obtained by Prof. Slesser, of Belfast, as an extension of the equations in Art. 107, which had been previously shown to him by the author.

It may be observed that these equations contain no quantities independent of the moving axes.

116. It will frequently be necessary to refer these moving axes to other axes fixed in space. Taking the same notation as in Art. 103, it is obvious (the axes being treated as a body consisting simply of three straight lines) that we shall obtain the results

$$\left.\begin{aligned} \frac{d\psi}{dt}\sin\theta &= -\theta_1\cos\phi + \theta_2\sin\phi \\ \frac{d\theta}{dt} &= \theta_1\sin\phi + \theta_2\cos\phi \\ \frac{d\psi}{dt}\cos\theta + \frac{d\phi}{dt} &= \theta_3 \end{aligned}\right\}.$$

These equations will determine θ_1, θ_2, θ_3 in terms of the arbitrary quantities θ, ϕ, ψ.

117. Ex. *A perfectly rough plane revolves uniformly about a vertical axis in its own plane, a sphere being placed in contact with the plane, rolls along it under the action of gravity. It is required to find the motion.*

Let the axis of revolution be taken as the axis of z, and let the axis of x be fixed in the plane and turn round the axis of z with an angular velocity n. Let a be the radius and M the mass of the sphere; F, F' the frictions between it and the plane resolved along the axes of x and z, and R the normal reaction. The equations of motion of the centre of gravity are by Art. 59

$$\frac{d^2x}{dt^2} - n^2x = \frac{F}{M} \qquad\text{(1)}$$

$$-an^2 + 2n\frac{dx}{dt} = \frac{R}{M} \qquad\text{(2)}$$

$$\frac{d^2z}{dt^2} = -g + \frac{F'}{M} \qquad\text{(3)}.$$

The equations of rotation by Art. 113 are

$$\frac{d\omega_x}{dt} - n\omega_y = -\frac{F'a}{A} \quad \text{.....................} (4)$$

$$\frac{d\omega_y}{dt} + n\omega_x = 0 \quad \text{..........................} (5)$$

$$\frac{d\omega_z}{dt} = \frac{Fa}{A} \quad \text{.........................} (6).$$

The geometrical equations, since the point in contact with the plane is at rest relatively to the plane, are

$$\frac{dx}{dt} + a\omega_z = 0 \quad \text{........................} (7)$$

$$\frac{dz}{dt} - a\omega_x = 0 \quad \text{........................} (8).$$

To solve these, we proceed thus. Substituting from (7) in (6) we get $F = \frac{-A}{a^2}\frac{d^2x}{dt^2}$. Hence by (1)

$$\frac{A + Ma^2}{Ma^2}\frac{d^2x}{dt^2} - n^2x = 0 \quad \text{..................} (9).$$

Let $\frac{Ma^2}{A + Ma^2} = \sin^2 i$. Then $\sin i = \sqrt{\frac{5}{7}}$. Solving equation (9) we get

$$x = \alpha \,.\, \epsilon^{\sin i\, nt} + \beta\epsilon^{-\sin i\, nt},$$

where α, β are arbitrary constants.

Again substituting from (8) in (4) and (5) we have

$$\left.\begin{aligned} \frac{d^2z}{dt^2} - an\omega_y &= -\frac{F'a^2}{A} \\ a\frac{d\omega_y}{dt} + n\frac{dz}{dt} &= 0 \end{aligned}\right\};$$

integrating this last,

$$a\omega_y + nz = \gamma;$$

$$\therefore\ A\frac{d^2z}{dt^2} + An^2z = A\gamma - F'a^2.$$

Hence from (3)

$$(A + Ma^2)\frac{d^2z}{dt^2} + An^2z = A\gamma - Mga^2$$
$$= - Ma^2g'.$$

To simplify the constants suppose the sphere to start from rest, and let the initial co-ordinates of the centre of gravity be $x = x_0$, $y = a$, $z = 0$.

Then $$\alpha = \beta = \frac{x_0}{2}, \text{ and } \gamma = 0.$$

Hence $$\frac{d^2z}{dt^2} + n^2\cos^2 i\,.\,z = -g\sin^2 i;$$

$$\therefore\ z = -\frac{g}{n^2}\tan^2 i\,\{1 - \cos(nt\cos i)\}.$$

Thus it appears that the sphere will not fall down. It will never descend more than $\frac{5g}{n^2}$ below its original position. If n be zero the above value of z becomes

$$z = -\frac{1}{2}g\sin^2 i\,.\,t^2.$$

118. Ex. *A sphere rolls under the action of gravity on a perfectly rough surface of revolution, placed with its axis of figure vertical. It is required to determine the motion.*

Let the moving axes of C, A and B be respectively the normal to the surface, a tangent to the meridian of the surface at the point of contact, and a perpendicular to both. Let the axis of figure of the surface be taken as the axis of z, and let any two fixed lines, at right angles, be taken as axes of x and y.

Let F, F' be the resolved parts of the friction along the axes of A and B; and R the normal reaction, a the radius of the sphere.

Let the axes of C and z make an angle θ with each other, and let ψ be the angle between the planes Cz and xz. Then clearly as in the second part of Art. 108,

$$\theta_1 = -\sin\theta\frac{d\psi}{dt}, \quad \theta_2 = \frac{d\theta}{dt}, \quad \theta_3 = \cos\theta\frac{d\psi}{dt}.$$

Hence the equations of Art. 115 become

$$\frac{d\omega_1}{dt} - \omega_2\cos\theta\frac{d\psi}{dt} + \omega_3\frac{d\theta}{dt} = \frac{F'a}{A} \quad \text{.................} \quad (1),$$

$$\frac{d\omega_2}{dt} + \omega_3\sin\theta\frac{d\psi}{dt} + \omega_1\cos\theta\frac{d\psi}{dt} = \frac{-Fa}{A} \quad \text{........} \quad (2),$$

$$\frac{d\omega_3}{dt} - \omega_1\frac{d\theta}{dt} - \omega_2\sin\theta\frac{d\psi}{dt} = 0 \quad \text{..................} \quad (3),$$

the mass of the sphere being taken as unity.

The equations of Art. 114 for the motion of the centre of gravity become, since $w = 0$,

$$\frac{du}{dt} - v\cos\theta\frac{d\psi}{dt} = g\sin\theta + F \quad \text{............} \quad (4),$$

$$\frac{dv}{dt} + u\cos\theta\frac{d\psi}{dt} = F' \quad \text{.......................} \quad (5),$$

$$-u\frac{d\theta}{dt} - v \,.\, \sin\theta\frac{d\psi}{dt} = R - g\cos\theta \quad \text{..........} \quad (6).$$

And the geometrical equations are

$$u - a\omega_2 = 0 \quad \text{........................} \quad (7),$$

$$v + a\omega_1 = 0 \quad \text{........................} \quad (8).$$

Also if $\rho - a$ be the radius of curvature of the meridian of the surface, r the distance of the centre of the sphere from the axis of z, we have

$$a\omega_2 = u = \rho \frac{d\theta}{dt} \text{ .... } (9),$$

$$-a\omega_1 = v = r \frac{d\psi}{dt} \text{ .... } (10).$$

To solve these equations.

Eliminating F, F', u, v, we get

$$(A + a^2)\left(\frac{d\omega_1}{dt} - \omega_2 \cos\theta \frac{d\psi}{dt}\right) + A\omega_3 \frac{d\theta}{dt} = 0 \text{ .... } (11),$$

$$(A + a^2)\left(\frac{d\omega_2}{dt} + \omega_1 \cos\theta \frac{d\psi}{dt}\right) + A\omega_3 \sin\theta \frac{d\psi}{dt} = ga\sin\theta \text{ ...} (12),$$

$$\frac{d\omega_3}{dt} - \omega_1 \frac{d\theta}{dt} - \omega_2 \sin\theta \frac{d\psi}{dt} = 0 \text{ .... } (13).$$

Multiplying these equations by ω_1, ω_2, $A\omega_3$, adding, and integrating, we get

$$(A + a^2)(\omega_1^2 + \omega_2^2) + A\omega_3^2 = \alpha + 2\int g\rho \sin\theta d\theta \text{ ...... } (14),$$

an equation which may also be obtained from the principle of vis viva.

Also by substituting from (9) and (10) in (13),

$$\frac{d\omega_3}{d\theta} = \omega_1\left(1 - \frac{\rho}{r}\sin\theta\right) \text{ .... } (15).$$

Again, substituting from (9) and (10) in (11) we have

$$(A + a^2)\left(\frac{d\omega_1}{d\theta} + \frac{\rho}{r}\cos\theta\ \omega_1\right) + A\omega_3 = 0 \text{ ...... } (16).$$

Differentiating this, and substituting from (15), we have

$$\frac{d^2\omega_1}{d\theta^2} + \frac{\rho}{r}\cos\theta \frac{d\omega_1}{d\theta} + P\omega_1 = 0,$$

where P is a known function of ρ, r, θ. Now ρ and r may be found from the equation to the meridian curve as functions of θ. Hence P is a known function of θ. Solving this equation by the ordinary rules, we have ω_1 expressed in terms of θ, and then by (14) and (15) ω_2 and ω_3 may be determined. Knowing ω_1 and ω_2, equations (9) and (10) will determine θ and ψ, and hence the motion of the sphere is found.

119. If the surface of revolution be a sphere, we have $\rho \sin\theta = r$, and hence by equation (15) ω_3 is constant. Equation (16) becomes in that case

$$\frac{d\omega_1}{d\theta} + \cot\theta \,.\, \omega_1 = -\frac{A}{A+a^2}\,\omega_3;$$

$$\therefore \sin\theta \,.\, \omega_1 = \beta + \frac{A\omega_3}{A+a^2} \,.\, \cos\theta.$$

The remainder of the solution is the same as before, and does not present any difficulty.

120. PROP. V. *To extend Euler's equations of motion to the case in which the shape and structure of the body are being gradually altered during the motion by changes of temperature or any other cause.*

Let x, y, z be the co-ordinates of any particle of mass m at the time t, referred to axes fixed in space. Then we have the equation of motion

$$\Sigma m\left(x\frac{d^2y}{dt} - y\frac{d^2x}{dt^2}\right) = N \;.............. \;(1),$$

and two similar equations.

Let $$h_3 = \Sigma m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) \;.................. \;(2),$$

with similar expressions for h_1, h_2.

Then the equation (1) becomes

$$\frac{dh_3}{dt} = N \;........................ \;(3).$$

Let the motion be referred to three rectangular axes Ox', Oy', Oz' moving in any manner about the origin O. Let α, β, γ be the angles these three axes make with the fixed axis of z.

Now h_3 is the sum of the products of the mass of each particle into twice the projection on the plane of xy of the area of the surface traced out by the radius vector of that particle drawn from the origin. Let h_1', h_2', h_3' be the corresponding "areas" described on the planes $x'y'$, $y'z'$, $z'x'$ respectively. Then by a known theorem proved in Geometry of Three Dimensions, the sum of the projections of h_1', h_2', h_3' on xy is equal to h_3;

$$\therefore\ h_3 = h_1' \cos\alpha + h_2' \cos\beta + h_3' \cos\gamma \ldots\ldots\ldots\ldots (4).$$

Since the fixed axes are quite arbitrary, let them be taken so that the moving axes are passing through them at the time t. Then

$$h_1' = h_1, \quad h_2' = h_2, \quad h_3' = h_3;$$

and by the same reasoning as in Arts. 114 and 115, we can deduce from equation (4) that

$$\frac{dh_3}{dt} = \frac{dh_3'}{dt} - h_1'\theta_2 + h_2'\theta_1 \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

where θ_1, θ_2, θ_3 are the angular velocities of the axes with reference to themselves.

Hence the equations of motion of the system become

$$\left.\begin{aligned} \frac{dh_1}{dt} - h_2\theta_3 + h_3\theta_2 &= L \\ \frac{dh_2}{dt} - h_3\theta_1 + h_1\theta_3 &= M \\ \frac{dh_3}{dt} - h_1\theta_2 + h_2\theta_1 &= N \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

These equations may be put under another form which is more convenient. Let x', y', z' be the co-ordinates of the particle m referred to the moving axes, and let

$$H_3 = \Sigma m \left(x' \frac{dy'}{dt} - y' \frac{dx'}{dt}\right).$$

Since the fixed axes coincide with these at the time t, we have

$$x = x', \quad y = y',$$

and by Art. 114,

$$\left.\begin{aligned} \frac{dx}{dt} &= \frac{dx'}{dt} + \theta_2 z - \theta_3 y \\ \frac{dy}{dt} &= \frac{dy'}{dt} + \theta_3 x - \theta_1 z \end{aligned}\right\};$$

$$\therefore\ h_3' = H_3 + C\theta_3 - E\theta_1 - D\theta_2;$$

and by similar reasoning,

$$h_1' = H_1 + A\theta_1 - F\theta_2 - E\theta_3,$$
$$h_2' = H_2 + B\theta_2 - D\theta_3 - F\theta_1.$$

Hence the general equation of motion becomes

$$\frac{d}{dt}(C\theta_3 - E\theta_1 - D\theta_2 + H_3) + F(\theta_2^2 - \theta_1^2)$$
$$+ (B - A)\theta_1\theta_2 + E\theta_2\theta_3 - D\theta_1\theta_3 + \theta_1 H_2 - \theta_2 H_1 = N \ldots\ldots (7),$$

and two similar equations.

Let the moving axes be so chosen as to coincide with the principal axes at the time t. Then $D = 0$, $E = 0$, $F = 0$, and these equations become*

$$\left.\begin{aligned} \frac{d}{dt}(A\theta_1 + H_1) - (B - C)\theta_2\theta_3 + \theta_2 H_3 - \theta_3 H_2 &= L \\ \frac{d}{dt}(B\theta_2 + H_2) - (C - A)\theta_3\theta_1 + \theta_3 H_1 - \theta_1 H_3 &= M \\ \frac{d}{dt}(C\theta_3 + H_3) - (A - B)\theta_1\theta_2 + \theta_1 H_2 - \theta_2 H_1 &= N \end{aligned}\right\} \cdots (8).$$

In these equations H_1, H_2, H_3 give the motions of the system *relative* to the moving axes. Thus if the body be a

* These equations are due to Liouville, and were published in his *Mathematical Journal* in 1858.

rigid body, and if ω_1, ω_2, ω_3 were the angular velocities about the axes, we have

$$H_3 = \Sigma m \left(x' \frac{dy'}{dt} - y' \frac{dx'}{dt}\right)$$

$$= C(\omega_3 - \theta_3) - E(\omega_1 - \theta_1) - D(\omega_2 - \theta_2)$$

by Art. 92.

Euler's equations and the equations of Art. 115 are included in these equations as particular cases. Thus, if the system be a rigid body, and if the moving axes be fixed in the body, $H_1 = 0$, $H_2 = 0$, $H_3 = 0$, and the equations become

$$A\frac{d\theta_1}{dt} - (B - C)\,\theta_2\theta_3 = L,$$

and two similar equations.

If every axis in the body at O be a principal axis, we get

$$\frac{d\omega_1}{dt} - \omega_2\theta_3 + \omega_3\theta_2 = \frac{L}{A},$$

and two similar equations.

121. If the motion be such that the system is always *symmetrical* about each of the three moving axes, we have $H_1 = 0$, $H_2 = 0$, $H_3 = 0$. The equations then become

$$\left.\begin{aligned} \frac{d}{dt}(A\theta_1) - (B - C)\,\theta_2\theta_3 = L \\ \frac{d}{dt}(B\theta_2) - (C - A)\,\theta_3\theta_1 = M \\ \frac{d}{dt}(C\theta_3) - (A - B)\,\theta_1\theta_2 = N \end{aligned}\right\} \text{.............. (9).}$$

122. Ex. *An ellipsoid whose centre is fixed contracts by cooling, and being set in motion in any manner is under the action of no forces. Determine the motion.* Liouville's *Journal.*

Since the principal diameters are principal axes at the fixed point, we may take them as moving axes of reference. Also since the body is symmetrical about the axes, we may use equations (9). Hence we have

$$\left.\begin{aligned} \frac{d}{dt}(A\theta_1) - (B-C)\,\theta_2\theta_3 &= 0 \\ \frac{d}{dt}(B\theta_2) - (C-A)\,\theta_3\theta_1 &= 0 \\ \frac{d}{dt}(C\theta_3) - (A-B)\,\theta_1\theta_2 &= 0 \end{aligned}\right\}.$$

Multiplying these equations respectively by $A\theta_1$, $B\theta_2$, $C\theta_3$, adding and integrating, we get

$$A^2\theta_1^2 + B^2\theta_2^2 + C^2\theta_3^2 = k^2,$$

where k is some constant.

To obtain another integral assume that

$$A = A_0 f(t), \quad B = B_0 f(t), \quad C = C_0 f(t).$$

Also let $\theta_1 f(t) = u_1$, $\theta_2 f(t) = u_2$, $\theta_3 f(t) = u_3$,

and $\dfrac{dT}{dt} = \dfrac{1}{f(t)}$, then the equations become

$$\left.\begin{aligned} A_0\frac{du_1}{dT} - (B_0 - C_0)\,u_2u_3 &= 0 \\ B_0\frac{du_2}{dT} - (C_0 - B_0)\,u_3u_1 &= 0 \\ C_0\frac{du_3}{dT} - (A_0 - B_0)\,u_1u_2 &= 0 \end{aligned}\right\};$$

and these equations will be integrated in the next section.

SECT. III. *The motion of a body of any form under the action of no external forces.*

123. PROP. *To determine the motion of a body about a fixed point, in the case in which there are no impressed forces.*

The equations of motion are

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} - (B-C)\,\omega_2\omega_3 = 0 \\ B\frac{d\omega_2}{dt} - (C-A)\,\omega_3\omega_1 = 0 \\ C\frac{d\omega_3}{dt} - (A-B)\,\omega_1\omega_2 = 0 \end{aligned}\right\};$$

multiplying these respectively by ω_1, ω_2, ω_3; adding and integrating we get

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = h^2 \quad\text{..................(1)},$$

where h^2 is an arbitrary constant.

Again, multiplying the equations respectively by $A\omega_1$, $B\omega_2$, $C\omega_3$ we get, similarly,

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = k^4 \quad\text{...............(2)},$$

where k^4 is an arbitrary constant.

These two first integrals may be deduced, as will hereafter be seen, from the principles of vis viva and Conservation of Areas.

To find a third integral, let

$$\omega_1^2 + \omega_2^2 + \omega_3^2 = \omega^2 \quad\text{.....................(3)};$$

$$\therefore\ \omega_1\frac{d\omega_1}{dt} + \omega_2\frac{d\omega_2}{dt} + \omega_3\frac{d\omega_3}{dt} = \omega\frac{d\omega}{dt};$$

then multiplying the original equations respectively by $\frac{\omega_1}{A}$, $\frac{\omega_2}{B}$, $\frac{\omega_3}{C}$, and adding we get

$$\omega \frac{d\omega}{dt} = \left(\frac{B-C}{A} + \frac{C-A}{B} + \frac{A-B}{C}\right) \omega_1 \omega_2 \omega_3 \quad \text{.........(4).}$$

But solving the equations (1), (2), (3), we get

$$\left.\begin{aligned} \omega_1^2 &= \frac{BC}{(A-C)(A-B)} \cdot (-\lambda_1 + \omega^2) \\ \omega_2^2 &= \frac{CA}{(B-A)(B-C)} \cdot (-\lambda_2 + \omega^2) \\ \omega_3^2 &= \frac{AB}{(C-B)(C-A)} \cdot (-\lambda_3 + \omega^2) \end{aligned}\right\} \quad \text{............(5),}$$

where $\lambda_1 = \dfrac{h^2(B+C) - k^4}{BC}$, with similar expressions for λ_2 and λ_3. Substituting in equation (4), we have

$$\omega \frac{d\omega}{dt} = \sqrt{(\lambda_1 - \omega^2)(\lambda_2 - \omega^2)(\lambda_3 - \omega^2)} \quad \text{...............(6),}$$

since $$\frac{BC(B-C) + CA(C-A) + AB(A-B)}{(B-C)(C-A)(A-B)} = -1.$$

The integration of equation (6) can be reduced without difficulty to depend on an elliptical integral. The integration can be effected in finite terms in two cases; when $A = B$, and when $k^4 = Bh^2$, where B is neither the greatest nor the least of the three quantities A, B, C. Both these cases will be discussed further on.

124. Let the momental ellipsoid at the fixed point be constructed, and let its equation be

$$Ax^2 + By^2 + Cz^2 = \epsilon^4 \quad \text{....................(7).}$$

Let r be the radius vector of the momental ellipsoid coincident with the instantaneous axis, and p the perpendicular from the centre on the tangent plane at the extremity of r. Also, as before, let ω be the angular velocity about the instantaneous axis.

Then since

$$\frac{\omega_1}{x}=\frac{\omega_2}{y}=\frac{\omega_3}{z}=\frac{\omega}{r},$$

comparing equations (1) and (7) we get

$$\frac{\omega^2}{r^2}=\frac{h^2}{\epsilon^4};$$

$$\therefore\ \omega=\frac{h}{\epsilon^2}.r \quad\text{.........................(8).}$$

Again, the expression for the perpendicular on the tangent plane at (x, y, z) is known to be

$$\frac{1}{p^2}=\frac{A^2x^2+B^2y^2+C^2z^2}{\epsilon^8},$$

hence, as before, comparing this equation with (2), we have

$$\frac{\omega^2}{r^2}=\frac{k^4p^2}{\epsilon^8};$$

$$\therefore\ \omega=\frac{k^2p}{\epsilon^4}.r.$$

Comparing this with (8) we see that

$$p=\frac{h\epsilon^2}{k^2} \quad\text{.........................(9).}$$

From these two equations we infer:

First. The angular velocity about the radius vector round which the body is turning varies as that radius vector.

Secondly. The perpendicular on the tangent plane at the extemity of the axis of revolution is constant, or, which is the same thing, the area of the section of the momental ellipsoid diametral to the axis of revolution is constant and equal to

$$\pi\frac{k^2\epsilon^4}{h\sqrt{ABC}}.$$

Thirdly. The angular velocity about p the perpendicular on the tangent plane is constant. For the cosine of the angle between p and r is $\frac{p}{r}$, hence the resolved angular velocity is $\omega \frac{p}{r} = \frac{h^2}{k^2}$ by (8) and (9).

125. It remains to determine the motion of the body in space. This may be deduced from equations (IV) but we may proceed more simply thus:

If the body be referred to axes fixed in space, then

$$\Sigma m \left(x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2} \right) = 0.$$

Integrating, we have

$$\Sigma m \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) = h_3$$

So also

$$\left. \begin{aligned} \Sigma m \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) &= h_3 \\ \Sigma m \left(y \frac{dz}{dt} - z \frac{dy}{dt} \right) &= h_1 \\ \Sigma m \left(z \frac{dx}{dt} - x \frac{dz}{dt} \right) &= h_2 \end{aligned} \right\},$$

h_1, h_2, h_3 being arbitrary constants to be determined from the initial conditions of motion.

Let the motion be referred to three co-ordinate axes Ox', Oy', Oz' moving in any manner about the fixed point O. Let a_1, b_1, c_1; a_2, b_2, c_2; a_3, b_3, c_3 be the direction-cosines of these with reference to the former system of axes. Also let h_1', h_2', h_3' be the quantities corresponding to h_1, h_2, h_3.

Now $h_3 = \Sigma m \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right)$ is the sum of the products of the mass of each particle into twice the projection on the plane of xy of the area of the surface traced out by the radius vector of that particle drawn from the origin.

Hence by a known theorem proved in Analytical Geometry of Three Dimensions, the sum of the projections of h_1', h_2', h_3' on the plane of xy is equal to h_3;

$$\therefore\ h_3 = c_1 h_1' + c_2 h_2' + c_3 h_3'.$$

Now $$h_3' = \Sigma m \left(x' \frac{dy'}{dt} - y' \frac{dx'}{dt} \right)$$

$$= \Sigma m \left(x'^2 + y'^2\right) \omega_{z'} - \Sigma m x'z' . \omega_{x'} - \Sigma m y'z' . \omega_{y'}.$$

Take the second system of axes so that the principal axes of the body coincide with them at the moment under consideration. Then

$$h_3' = C\omega_3.$$

Hence the above equation becomes

$$h_3 = A\omega_1 . c_1 + B\omega_2 . c_2 + C\omega_3 . c_3.$$

So also $$h_1 = A\omega_1 . a_1 + B\omega_2 . a_2 + C\omega_3 . a_3,$$

$$h_2 = A\omega_1 . b_1 + B\omega_2 . b_2 + C\omega_3 . b_3.$$

The straight line whose direction cosines are proportional to h_1, h_2, h_3 is clearly fixed throughout the motion. It is therefore called the "Invariable Line."

Let α, β, γ be the angles made by the invariable line with the principal axes. Then

$$\cos\alpha = \frac{h_1 a_1 + h_2 b_1 + h_3 c_1}{\sqrt{h_1^2 + h_2^2 + h_3^2}}.$$

Substituting and remembering the equations

$$a_1^2 + b_1^2 + c_1^2 = 1,$$

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0,$$

with four other similar equations, we get

$$\cos\alpha = \frac{A\omega_1}{k^2}.$$

Similarly,

$$\cos\beta = \frac{B\omega_2}{k^2}, \text{ and } \cos\gamma = \frac{C\omega_3}{k^2}.$$

These are the direction-cosines of a line fixed in space referred to axes fixed in the body and moving with it.

The direction-cosines of the instantaneous axis are

$$\frac{\omega_1}{\omega}, \quad \frac{\omega_2}{\omega}, \quad \frac{\omega_3}{\omega};$$

hence this axis meets the momental ellipsoid in a point whose co-ordinates are

$$\frac{\epsilon^2}{h}\omega_1, \quad \frac{\epsilon^2}{h}\omega_2, \quad \frac{\epsilon^2}{h}\omega_3.$$

The equation to the tangent plane at this point is therefore

$$A\omega_1 x + B\omega_2 y + C\omega_3 z = \epsilon^2 h;$$

hence it is perpendicular to the invariable line. Also as its distance from the fixed point is constant, this tangent plane is absolutely fixed in space.

The motion of the momental ellipsoid is therefore such that, its centre being fixed, it always touches a fixed plane, and the point of contact being in the instantaneous axis has no velocity. Hence the motion may be represented by supposing the central ellipsoid to *roll* on the fixed plane, with its centre fixed. This plane is called the "Invariable tangent Plane."

126. The motion of the ellipsoid has been represented by supposing it to roll on a certain plane, the centre of the ellipsoid being supposed fixed. The point of contact is the extremity of the instantaneous axis, which therefore traces out two curves, one on the surface of the ellipsoid which is fixed in the body, and the other on the plane which is fixed in space. The first of these curves is called the *polhode*, the second the *herpolhode.*

This same motion may be also represented geometrically in another manner. If the extremity of a radius vector from the fixed point trace out the polhode, the radius vector itself traces out a conical surface fixed in the body which is of the second order. If the extremity of a radius vector traces out the herpolhode, the radius vector itself traces out a conical surface fixed in space. If we suppose the first of these surfaces to *roll* on the second, the motion of the body will be truly represented.

These representations of the motion are due to Poinsot.

127. If a body, having one point fixed, be set in rotation about any axis, it will not in general continue to rotate about it, but the axis will describe on the central ellipsoid the polhode passing through its initial position. It is also evident that it will move along this polhode always in the same direction. The equations to any polhode may be found from the consideration that the length of the perpendicular on the tangent plane at any point of the polhode is constant. Hence by Art. 124 its equations are

$$\left.\begin{array}{r} A^2x^2 + B^2y^2 + C^2z^2 = \lambda^6 \\ Ax^2 + By^2 + Cz^2 = \epsilon^4 \end{array}\right\},$$

where λ is the variable parameter. Eliminating y we have

$$A(A-B)x^2 + C(C-B)z^2 = \lambda^6 - \epsilon^4 B.$$

Hence if B be the axis of greatest or least moment of inertia, the signs of the coefficients of x^2 and z^2 will be the same, and the projection of the polhode will be an ellipse. But if B be the axis of mean moment of inertia, the projection is an hyperbola.

It follows, therefore, that all the polhodes are closed curves drawn round the axes of greatest and least moment. The boundary line which separates the two sets of polhodes is that polhode whose projection on the plane perpendicular to the axis of mean moment consists of an hyperbola whose concavity tends neither to the axis of greatest nor to the axis of least moment. In this case the projection consists of two straight lines whose equations are

$$A(A-B)x^2 - C(B-C)z^2 = 0.$$

Hence the polhode consists of two ellipses passing through the axis of mean moment, and therefore it corresponds to the case in which the perpendicular on the tangent plane is equal to the mean axis of the ellipsoid. This polhode is called the "separating polhode."

128. It is clear that the extremity of the axis of revolution will describe a closed curve round the axis of greatest or least moment according as it is initially on the one side or the other of the separating polhode. But in no case can it describe a closed curve about the axis of mean moment. This is usually expressed by saying that the rotations about the axes of greatest and least moment are *stable*, while that about the mean axis is *unstable*. These expressions are not however perfectly accurate, for the projection of the polhode on the plane of xy being

$$A(A-C)x^2 + B(B-C)y^2 = \lambda^6 - C\epsilon^4,$$

if the quantities $A(A-C)$, $B(B-C)$ should differ very much from each other, the polhode will be an elongated oval, and though the axis might have been originally only very slightly displaced from the principal axis, it will recede very far from it. So again if the extremity of the axis of revolution be placed on the separating polhode it is possible that it may tend continually to approach nearer to coincidence with the principal axis of mean moment. For, let B be the axis of mean moment, and A the axis of greatest moment, then

$$A(A-B)x^2 = C(B-C)z^2;$$

but since x and z are proportional to ω_1, ω_3, this becomes

$$A(A-B)\omega_1^2 = C(B-C)\omega_3^2.$$

But $$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = k^4;$$

therefore

$$\left.\begin{aligned} AB(A-C)\omega_1^2 &= (B-C)(k^4 - B^2\omega_2^2) \\ BC(A-C)\omega_3^2 &= (A-B)(k^4 - B^2\omega_2^2) \end{aligned}\right\} \cdots\cdots (1).$$

But $$B\frac{d\omega_2}{dt} - (C-A)\,\omega_1\omega_3 = 0;$$

$$\therefore\ B^2\frac{d\omega_2}{dt} = \mp\sqrt{\frac{(A-B)(B-C)}{AC}}\,.\,(k^4 - B^2\omega_2^2);$$

when ω_1, ω_3 have like signs $(C-A)\,\omega_1\omega_3$ is negative, and therefore $\frac{d\omega_2}{dt}$ must be negative, hence in this expression the upper or lower sign is to be used according as ω_1, ω_3 have like or unlike signs.

$$\therefore\ \frac{B}{k^4 - B^2\omega_2^2}\frac{d\omega_2}{dt} = \mp\frac{1}{B}\sqrt{\frac{(A-B)(B-C)}{AC}}$$

$$= \mp n \text{ suppose};$$

$$\therefore\ \frac{k^2 + B\omega_2}{k^2 - B\omega_2} = \alpha\,.\,e^{\mp 2k^2nt} \quad\text{.................. (2).}$$

Hence as t is indefinitely increased ω_2 approaches to $\mp\frac{k^2}{B}$ as its limit, and therefore by (1) ω_1 and ω_3 approach zero.

The conclusion therefore is that the axis of revolution continually approaches to coincidence with the mean axis of principal moment, but never actually coincides with it. It approaches the nearest end of the mean axis when ω_1, ω_3 have *unlike* signs.

129. Prop. *To find the form of the herpolhode.*

Let s represent the arc of the polhode, and σ that of the herpolhode. Then since one curve rolls on the other

$$s = \sigma \quad\text{.............................. (1).}$$

Let r be the radius vector of the polhode measured from the centre of the ellipsoid, and ρ that of the herpolhode measured from the foot of the perpendicular drawn from the centre of the ellipsoid on the fixed plane, containing the herpolhode; then if f be this perpendicular

$$r^2 = f^2 + \rho^2 \quad\text{........................ (2).}$$

Let the equation to the polhode be expressed in the form

$$\phi(s, r) = 0 \text{ .... } (3),$$

then the equation to the herpolhode is

$$\phi(\sigma, \sqrt{f^2 + \rho^2}) = 0 \text{ .... } (4).$$

130. Since the projection of the polhode on one of the principal planes is always an ellipse, the polhode must be a re-entering curve.

By considering the herpolhode to be traced out by the rolling of an ellipsoid on the plane of the paper, it is clear that the herpolhode always lies between two circles which it alternately touches. The herpolhode is therefore not in general re-entering; but if the angular distance of the two points in which it successively touches the same circle be commensurable with 2π, it will be a re-entering curve.

131. The equation to the herpolhode cannot generally be found. If however the polhode be the separating polhode, the integrations can be effected and the herpolhode can be found. The polhode is in this case a plane curve and therefore an ellipse. Let β, b, be the semi-axes of the ellipse. Then $1 - \left(\frac{dr}{ds}\right)^2$ is the square of the sine of the angle between any radius vector r and the perpendicular p on the tangent. Let r' be the semi-diameter conjugate to r. Then

$$1 - \left(\frac{dr}{ds}\right)^2 = \frac{p^2}{r^2} = \frac{b^2\beta^2}{r^2 r'^2}$$

$$= \frac{b^2\beta^2}{r^2(\beta^2 + b^2 - r^2)},$$

but $r^2 = \rho^2 + b^2$; therefore $rdr = \rho d\rho$;

$$\therefore \left(\frac{d\rho}{ds}\right)^2 = \frac{\beta^2 - b^2 - \rho^2}{\beta^2 - \rho^2},$$

but $(ds)^2 = (d\rho)^2 + \rho^2 (d\theta)^2$, where θ is the vectorial angle, hence

$$d\theta = \frac{b d\rho}{\rho\sqrt{n^2 - \rho^2}},$$

where $n^2 = \beta^2 - b^2$. Integrating we have

$$\theta = \frac{b}{n} \cdot \log \frac{\rho}{n + \sqrt{n^2 - \rho^2}};$$

$$\therefore \frac{2n}{\rho} = e^{\frac{n\theta}{b}} + e^{-\frac{n\theta}{b}},$$

the prime radius being drawn to the point where the plane of the herpolhode is met by the instantaneous axis $r = \beta$.

This is the equation to the herpolhode. It remains to find b and β in terms of A, B, C. Since β is a diameter of the section of the ellipsoid made by the plane containing the axes A and C, and is such that the perpendicular on the tangent at its extremity $= b$, we have

$$b^2 (a^2 + c^2 - \beta^2) = a^2 c^2,$$

where a and c are the semi-axes of the section;

$$\therefore n^2 = \beta^2 - b^2 = \frac{(b^2 - a^2)(c^2 - b^2)}{b^2}$$

after some reduction. But $a^2 = \frac{\epsilon^4}{A}$, $b^2 = \frac{\epsilon^4}{B}$, $c^2 = \frac{\epsilon^4}{C}$,

hence

$$\frac{n^2}{\epsilon^4} = \frac{(A - B) \cdot (B - C)}{A \cdot B \cdot C},$$

and the equation to the herpolhode becomes

$$\frac{2n}{\rho} = e^{g\theta} + e^{-g\theta},$$

where
$$g = \frac{n\sqrt{B}}{\epsilon^2}.$$

This is the equation to a double spiral whose two branches form an infinite number of revolutions in opposite directions about the origin, and approach it without limit.

132. Prop. *To find the velocity with which the extremity of the instantaneous axis traverses the polhode or herpolhode.*

Let $v = \frac{ds}{dt} = \frac{d\sigma}{dt}$ be the velocity with which the instantaneous axis traverses either curve, and ω the angular velocity of the body about the instantaneous axis. Let R, R' be the principal radii of curvature at a common point of the polhode and herpolhode of the two conical surfaces traced out in the body and in space by the instantaneous axis. These surfaces may be supposed to be made up of triangular planes. Let $d\varepsilon$ and $d\varepsilon'$ be the inclinations of two successive planes. Then the body in turning round the instantaneous axis describes the angle $d\varepsilon \pm d\varepsilon'$. But it also describes ωdt;

$$\therefore \ \omega dt = d\varepsilon \pm d\varepsilon'.$$

But
$$d\varepsilon = \frac{ds}{R}, \qquad d\varepsilon' = \frac{d\sigma}{R'};$$

$$\therefore \ \omega \quad \frac{ds}{dt}\left(\frac{1}{R} \pm \frac{1}{R'}\right),$$

the upper or lower sign to be used according as the curvatures are in opposite or the same directions.

As an example of the utility of this formula let us consider the case of the earth set in rotation about an axis making an angle α with the axis of figure. Then both the polhode and herpolhode are circles. Let c be the radius vector of the spheroid which is the initial axis of revolution.

Then by Meunier's theorem in Geometry of Three Dimensions,

$$R \cos \alpha = \text{radius of polhode}$$
$$= c \sin \alpha,$$

$$R' = \text{radius of herpolhode}$$
$$= c \sin \beta;$$

where β is the angle which the instantaneous axis makes with the invariable line,

$$\text{and} \quad \omega = \frac{2\pi}{24 \times 60 \times 60},$$

whence v may be found by the theorem.

If T, T' be the times in which the axis of revolution describes complete circles in the Earth and in space respectively, then

$$T = \frac{2\pi c \sin \alpha}{v}, \quad T' = \frac{2\pi c \sin \beta}{v}.$$

133. The momental ellipsoid is the reciprocal surface of the ellipsoid of gyration. Every curve on the one surface has its reciprocal curve on the other. The reciprocal curves of the polhode corresponding to any value of p, the constant perpendicular on the tangent plane, are the curves of intersection of the ellipsoid of gyration with the sphere whose radius is $r = \frac{\epsilon^2}{p}$. The reciprocal curves of the separating polhode are the central circular sections. The invariable line though fixed in space moves in the body, its intersection with the ellipsoid of gyration traces out the reciprocal of the polhode described by the instantaneous axis. The invariable line thus describes in the body a right cone on an elliptic base. This cone becomes a plane when the instantaneous axis describes the separating polhode. Hence in this case the body moves so that a certain plane fixed in the body always passes through a straight line fixed in space.

134. It is well known that the steadiness or stability of a moving body is much increased by a rapid rotation about a principal axis. The reason of this is as follows. The instantaneous axis describes a polhode in the body and a herpolhode in space. If the body be set rotating about an axis very near the principal axis of greatest or least moment, both the polhode and herpolhode will generally be very small

curves, and the direction of that principal axis of the body will be very nearly fixed in space. If now a small impulse f act on the body, the effect will be to alter slightly the position of the instantaneous axis. It will be moved from one polhode to another very near the former, and thus the angular position of the axis in space will not be much affected. Let Ω be the angular velocity of the body, ω that generated by the impulse, then by the parallelogram of angular velocities, the change in the position of the instantaneous axis cannot be greater than $\sin^{-1}\dfrac{\omega}{\Omega}$. If therefore Ω be great, ω must also be great, to produce any considerable change in the axis of rotation. But if the body had no initial rotation Ω, the impulse may generate an angular velocity ω about an axis not nearly coincident with a principal axis. Both the polhode and the herpolhode may then be large curves, and the instantaneous axis of rotation will move about both in the body and in space. The motion will then appear very unsteady. In this manner, for example, we may explain why in the game of cup and ball, spinning the ball about a vertical axis makes it more easy to catch on the spike. Any motion caused by a wrong pull of the string or by gravity, will not produce so great a change of motion as it would have done if the ball had been initially at rest. The fixed direction of the earth's axis in space is also due to its rotation about its axis of figure.

135. When the invariable line is taken as the axis of z the equations of motion may be put under another form.

The direction-cosines of the invariable line are

$$\cos\alpha = \frac{A\omega_1}{k^2}, \quad \cos\beta = \frac{B\omega_2}{k^2}, \quad \cos\gamma = \frac{C\omega_3}{k^2}.$$

Referring to the figure of Art. 103, we have

$$\begin{aligned}
\cos\alpha &= \cos AE \,.\, \cos EZ = -\cos\phi \,.\, \sin\theta \\
\cos\beta &= \cos BE \,.\, \cos EZ = \sin\phi \,.\, \sin\theta \\
\cos\gamma &= \cos CZ \qquad\qquad\;\; = \cos\theta.
\end{aligned}$$

Equating these values of $\cos\alpha$, $\cos\beta$, $\cos\gamma$ we get ω_1, ω_2, ω_3 in terms of θ and ϕ. Then substituting in the geometrical equations of Art. 103, we have

$$\left.\begin{aligned} \frac{d\theta}{dt} &= -k^2 \sin\theta \sin\phi \cos\phi \left(\frac{1}{A} - \frac{1}{B}\right) \\ \sin\theta \frac{d\psi}{dt} &= k^2 \sin\theta \left(\frac{\cos^2\phi}{A} + \frac{\sin^2\phi}{B}\right) \\ \frac{d\phi}{dt} + \cos\theta \frac{d\psi}{dt} &= \frac{k^2\cos\theta}{C} \end{aligned}\right\}.$$

If $A = B$, these equations will be greatly simplified. We have in that case

$$\left.\begin{aligned} \frac{d\theta}{dt} &= 0 \\ \frac{d\psi}{dt} &= \frac{k^2}{A} \\ \frac{d\phi}{dt} + \cos\theta \frac{d\psi}{dt} &= \frac{k^2\cos\theta}{C} \end{aligned}\right\}.$$

The results of the next Article may be easily deduced from these equations.

136. Prop. *To determine the motion of the body when two of the principal moments at the fixed point are equal.*

Let the body be set rotating with an angular velocity ω about an instantaneous axis OI, making an angle α with OC the axis of figure.

The momental ellipsoid becomes in this case a spheroid, the axis of which is the axis of the body. From the symmetry of the figure it is evident that as the spheroid rolls on the invariable plane, the angles LOC, LOI are *constant*, and that the three axes OI, OL, OC are always in one plane.

The section of the momental ellipsoid by the plane in which OI, OL, OC lie is an ellipse whose semi-axes are $\sqrt{\frac{\epsilon^4}{A}}$ and $\sqrt{\frac{\epsilon^4}{C}}$. Also, OL being perpendicular to the tangent at I to this ellipse, is perpendicular to the conjugate diameter of OI; $\therefore$ by conics

$$\tan\beta = \frac{A}{C}\tan\alpha.$$

The angular velocity of the body about OI varies as the radius vector OI of the spheroid, and is therefore constant. Hence OI describes a right cone in the body round OC with a uniform angular velocity, and a right cone in space round OL with a uniform angular velocity.

The angular velocity ν of OI round OC in the body may be found most readily by referring to the original equations of motion in Art. 123. We have in this case

$$\left.\begin{aligned} \frac{d\omega_1}{dt} - n\frac{A-C}{A}\omega_2 = 0 \\ \frac{d\omega_2}{dt} + n\frac{A-C}{A}\omega_1 = 0 \end{aligned}\right\}.$$

Solving these in the usual way we have

$$\left.\begin{aligned} \omega_1 = F\cos\left(\frac{A-C}{A}nt + f\right) \\ \omega_2 = F\sin\left(\frac{A-C}{A}nt + f\right) \end{aligned}\right\},$$

where F and f are arbitrary constants. Let χ be the angle the projection of the instantaneous axis on the plane perpendicular to OC makes with the fixed straight line which has been taken for the axis OA, then

$$\tan\chi = \frac{\omega_2}{\omega_1}, \quad \text{and } \nu = \frac{d\chi}{dt};$$

$$\therefore \chi = \frac{A-C}{A}nt + f;$$

$$\therefore \ \nu = \frac{A - C}{A} n,$$

where $n = \omega \cos \alpha$ is the angular velocity about the axis of figure.

The angular velocity ν' of OI round OL in space may be found from the consideration that OC, OI, OL are always in one plane. Describe a sphere round O as centre cutting OC, OL in C and L. The displacement CC' of C in the time dt due to the angular velocity ω round I is $\omega \sin \alpha dt$. Hence $\omega \dfrac{\sin \alpha}{\sin \beta} dt$ is the angle made by the two arcs CL, $C'L$ on the sphere. But, since OC, OI, OL are always in one plane, this is the angular velocity of OI about OL. Hence

$$\nu' = \omega \frac{\sin \alpha}{\sin \beta}$$

$$= \omega \frac{\sqrt{A^2 \sin^2 \alpha + C^2 \cos^2 \alpha}}{A}.$$

SECT. IV. *Small Oscillations in Three Dimensions about a position of equilibrium.*

137. Problems of small oscillations are usually of two kinds. The mean position about which the oscillation takes place may either be fixed or it may have a motion in space. These will be considered in turn. As an example of the former, we may have a body in equilibrium rotating about an axis through its centre of gravity and under any conditions of constraint. Suppose the body to be slightly disturbed, we have first to determine whether it will or will not remain near its position of rest. One condition will usually be that the axis of rotation must be very near to a principal axis in the body, for no other axis could be one of free permanent rotation. We have also to determine the motion of the body about its position of rest. This will be determined when we know at every instant the position of a line fixed in the body and the angular velocity about it.

There are two simplifications which when they occur will be of the greatest assistance, *first*, the angular velocity about the principal axis which was very near to the instantaneous axis of rotation at the beginning of the motion may be constant, and *secondly*, when gravity is the only force, the altitude of the centre of gravity above some horizontal fixed plane may be constant. A little consideration will show that these will be true in most cases.

The general problem to be considered in this section may be stated as follows:

PROB. *A body has one point fixed in space and is in rotation about an instantaneous axis which makes a small angle with a principal axis in the body at the fixed point. Supposing the body to be making small oscillations about its mean position, it is required to determine the motion.*

138. *First method.* Let O be the fixed point, OA, OB, OC the principal axes at that point, and let OC be that principal axis from which the instantaneous axis is never far distant. Let the mean position of OC be taken as the axis of z.

By hypothesis L, M, N, the moments of the forces about the axes are all small quantities, also ω_1, ω_2 are small. Let n be the mean value of ω_3, then in the small terms we may put $\omega_3 = n$. Hence the equations of motion become

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} - (B-C)\,n\omega_2 &= L \\ B\frac{d\omega_2}{dt} - (C-A)\,n\omega_1 &= M \\ C\frac{d\omega_3}{dt} \qquad\qquad\qquad &= N \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Let the position of the axis OC in space be defined as usual by the angles θ, ϕ, ψ, and let

$$\left.\begin{aligned} p &= \sin\theta\cos\phi \\ q &= -\sin\theta\sin\phi \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

then p and q are both small quantities.

Referring to the figure and reasoning of Art. 103, we have

$$\frac{d\phi}{dt} + \cos\theta\,\frac{d\psi}{dt} = \omega_3,$$

$$\omega_1 = \frac{d\theta}{dt}\sin\phi - \sin\theta\,\frac{d\psi}{dt}\cos\phi,$$

$$\therefore\ \omega_1\cos\theta = \cos\theta\frac{d\theta}{dt}\sin\phi - \sin\theta\cos\phi\left(\omega_3 - \frac{d\phi}{dt}\right),$$

similarly,

$$\left.\begin{aligned} &= -\frac{dq}{dt} - \omega_3 p, \\ \omega_2\cos\theta &= \frac{dp}{dt} - \omega_3 q \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots (3).$$

Also $-p$ and $-q$ are the direction-cosines of OZ with reference to the axes OA and OB, for

$$-p = -\sin\theta\cos\phi = \cos ZE\,.\,\cos AE = \cos AZ.$$

Similarly, $\qquad -q = \cos BZ.$

But since OC very nearly coincides with OZ, θ is very small and the above equations become

$$\left.\begin{aligned} \frac{d\phi}{dt} + \frac{d\psi}{dt} &= \omega_3 \\ \omega_1 &= -\frac{dq}{dt} - np \\ \omega_2 &= \frac{dp}{dt} - nq \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Let the axis Ox be so chosen that the mean value of $\phi + \psi = nt$. Let two axes Ox', Oy' revolve round Oz with an angular velocity n, so that the angle $xOx' = nt$. Then these moving axes will never deviate far from the principal axes OA, OB. Also p and q will now be the direction-cosines of OC referred to these moving axes Ox', Oy'. For the points C and A lie very nearly in the arc joining Zx', hence since Zx' and CA are both right angles, the arcs ZA and Cx' are supplements, and $\therefore\ p = -\cos ZA = \cos Cx'$. Similarly, $q = \cos Cy'$.

Substituting for ω_1 and ω_2 from (4) in the equations of motion we get

$$\left.\begin{aligned} A\frac{d^2q}{dt^2}+(A+B-C)\,n\frac{dp}{dt}-(B-C)\,n^2q&=-L\\ B\frac{d^2p}{dt^2}-(A+B-C)\,n\frac{dq}{dt}-(A-C)\,n^2p&=M\\ C\frac{d\omega_3}{dt}&=N \end{aligned}\right\}\ \text{(A).}$$

These equations are linear, and when solved will determine the motion of OC with reference to the moving axes Ox', Oy'. The motion of OC referred to any *fixed* axes in space can then be easily found.

139. The quantities L and M are the moments about the axes OA and OB, and by the geometry peculiar to the proposed question must be expressed in terms of p and q. Since the squares of p and q are to be neglected, these expressions will be of the form

$$\left.\begin{aligned} L&=ap+a'q\\ M&=bp+b'q \end{aligned}\right\}.$$

If it be difficult to find the moments of the forces about OA, OB, we may find the moments about the axes Ox', Oy', Oz'. Let these be L', M', N'; then

$$\begin{aligned} L&=L'\cos Ax'+M'\cos Ay'+N'\cos Az'\\ &=L', \end{aligned}$$

since M', N' are small quantities and the angles Ay', Az' are nearly right angles.

Similarly $M=M'$, $N=N'$. Thus the *moments of the forces are the same whether they be taken about the principal axes OA, OB, OC, or about the co-ordinate axes Ox', Oy', Oz'.*

In finding these moments the following remark will also be useful. If p', q', 1 be the direction-cosines of any line OP near OC referred to the axes OA, OB, OC, then its direction-cosines referred to Ox', Oy', Oz' will be respectively

$$p+p',\ q+q',\ \text{and } 1.$$

For join CA by an arc of a great circle, and drop perpendiculars PN, $x'n$ on it from P and x'. Then since $\angle PN$ and $\angle x'n$ are very small, $\cos \angle Px'$ and $\cos \angle Nn$ only differ by quantities of the second order, hence

$$\begin{aligned}\cos Px' &= \cos Nn \\ &= \sin (Cn + NA) \\ &= \sin Cn \cos NA + \cos Cn \sin NA.\end{aligned}$$

Now Cn and AN are nearly right angles, and when multiplied by small terms their sines may be taken equal to unity. Also $\cos NA = \cos PA = p'$, and $\cos Cn = \cos Cx' = p$ when we reject the squares of small quantities. Hence we have

$$\cos Px' = p + p',$$

and similarly

$$\cos Py' = q + q'.$$

140. The equations for the determination of p and q will in general be of the form

$$\left.\begin{aligned}\frac{d^2q}{dt^2} + a\frac{dp}{dt} &= bp + cq \\ \frac{d^2p}{dt^2} - a'\frac{dq}{dt} &= b'q + c'p\end{aligned}\right\} \dots\dots\dots\dots\dots (1),$$

and they must be solved by the methods described in treatises on Differential Equations. For convenience of reference, we shall here give a short summary of the different steps.

Case I. It will frequently happen that the terms bp, $b'q$ are absent. When this is the case, assume

$$\left.\begin{aligned}p &= F\cos(\lambda t + f) \\ q &= G\sin(\lambda t + f)\end{aligned}\right\} \dots\dots\dots\dots\dots\dots (2),$$

substituting, we get

$$\left.\begin{aligned}G(\lambda^2 + c) &= -F \,.\, a\lambda \\ F(\lambda^2 + c') &= -Ga'\lambda\end{aligned}\right\} \dots\dots\dots\dots\dots\dots (3),$$

whence we have the biquadratic

$$(\lambda^2+c)(\lambda^2+c')=aa'\lambda^2 \quad\dots\dots\dots\dots\dots\quad (4),$$

for the determination of λ while either of equations (3) will give the ratio $\frac{G}{F}$.

Supposing the four values of λ to be all real, as will generally be the case, and equal to $\pm\lambda_1, \pm\lambda_2$, then the complete integral will be

$$p = F\cos(\lambda_1 t+f) + F'\cos(\lambda_2 t+f')$$

$$q = F\frac{a\lambda_1}{\lambda_1^2+c}\sin(\lambda_1 t+f) + F'\frac{a\lambda_2}{\lambda_2^2+c}\sin(\lambda_2 t+f'),$$

where F, F', f, f', are four arbitrary constants to be determined by the initial values of $p, q, \frac{dp}{dt}, \frac{dq}{dt}$.

If the four roots of the equation (4) be not all real, the expressions for p and q must be rationalized by writing for the sine and cosine their exponential values.

The equation (4) may be written in the form

$$\lambda^4+(c+c'-aa')\lambda^2+cc'=0.$$

In order that these values of λ^2 may be *real* we must have

$$(c+c'-aa')^2-4cc' = \textit{a positive quantity} \quad\dots\dots\dots\quad (A).$$

In order that the two values of λ^2 may have the *sam sign* we must have the last term of the quadratic positive,

$$\therefore\ cc' = \textit{a positive quantity} \quad\dots\dots\dots\dots\dots\quad (B).$$

In order that the values of λ^2 may both be *positive*, we must have the second term of the quadratic negative;

$$\therefore\ c+c'-aa' = \textit{a negative quantity} \quad\dots\dots\dots\dots\quad (C).$$

The condition (A) is satisfied if $c+c'$ and aa' have opposite signs. If c, c' be both negative and aa' positive, all three conditions are satisfied.

141. *Case II.* If the equations be complete, assume

$$\left.\begin{aligned} p &= F\epsilon^{\lambda t} \\ q &= G\epsilon^{\lambda t} \end{aligned}\right\} \qquad (2),$$

substituting, we get

$$\left.\begin{aligned} G(\lambda^2 - c) &= F(b - a\lambda) \\ F(\lambda^2 - c') &= G(b' + a'\lambda) \end{aligned}\right\} \qquad (3),$$

whence we have the biquadratic

$$(\lambda^2 - c)(\lambda^2 - c') = (b - a\lambda)(b' + a'\lambda) \qquad (4)$$

for the determination of λ while either of the equations (3) will give the ratio $\dfrac{G}{F}$.

If all the roots of this equation be real, the motion will not be oscillatory, a positive root will correspond to a motion that continually increases, a negative root to a motion that gradually dies away.

A pair of negative roots $\lambda = \alpha \pm \beta\sqrt{-1}$ corresponds to two terms in (2) of the form

$$\left.\begin{aligned} p &= (F+F')\,\epsilon^{\alpha t}\cos\beta t + (F-F')\sqrt{-1}\,\epsilon^{\alpha t}\sin\beta t \\ q &= (G+G')\,\epsilon^{\alpha t}\cos\beta t + (G-G')\sqrt{-1}\,\epsilon^{\alpha t}\sin\beta t \end{aligned}\right\}$$

and imaginary values must be given to F, F' and G, G' to render these expressions real.

From equation (3) we have

$$\frac{G}{F} = \frac{b - a\lambda}{\lambda^2 - c} = u + v\sqrt{-1},$$

where u and v are known, the value of $\lambda = \alpha + \beta\sqrt{-1}$ being substituted from equation (4). Hence clearly when we write $\lambda = \alpha - \beta\sqrt{-1}$, we have

$$\frac{G'}{F'} = u - v\sqrt{-1}.$$

Therefore if the first equation be written in the form

$$p = H\epsilon^{\alpha t}\cos\beta t + K\epsilon^{\alpha t}\sin\beta t,$$

the second will take the form

$$q = (Hu + Kv)\,\epsilon^{\alpha t}\cos\beta t + (Ku - Hv)\,\epsilon^{\alpha t}\sin\beta t.$$

It is evident that unless α be either zero or negative, the motion cannot be considered one of small oscillation.

The equation (4) may be written in the form

$$\lambda^4 - (c + c' - aa')\,\lambda^2 + (ab' - a'b)\,\lambda + cc' - bb' = 0.$$

In order that the four roots of this equation may be of the form $\lambda = \pm\beta\sqrt{-1}$, we must have

$$ab' - a'b = 0 \text{ .... (A).}$$

Then as before, in order that the two values of λ^2 may be real, both of the same sign, and that sign negative, we must have

$$\left.\begin{array}{c} (c + c' - aa')^2 - 4(cc' - bb') \\ cc' - bb' \\ c + c' - aa' \end{array}\right\} \text{all positive ... (B).}$$

142. Ex. *A conical top has its vertex resting on a perfectly rough horizontal plane, and is in rotation with its axis of figure very nearly vertical. Find the least angular velocity that it may not tumble down.*

The only force, besides those at the fixed point, which acts on the body is gravity, and if h be the distance of the centre of gravity from the vertex, the moments of these about the axes are

$$L = yZ - zY = -hqg,$$
$$M = zX - xZ = hpg.$$

Hence the equations of motion in Art. 138 become

$$A\frac{d^2q}{dt^2} + (2A - C)\,n\frac{dp}{dt} - (A - C)\,n^2q = hg\,.\,q,$$

$$A\frac{d^2p}{dt^2} - (2A - C)\,n\frac{dq}{dt} - (A - C)\,n^2p = hg\,.\,p,$$

taking the mass of the body to be unity.

These equations may be written in the form

$$A\frac{d^2q}{dt^2} + a\frac{dp}{dt} = bq,$$

$$A\frac{d^2p}{dt^2} - a\frac{dq}{dt} = bp.$$

To solve these, assume

$$\left.\begin{array}{l} p = F\cos(\lambda t + f) \\ q = G\sin(\lambda t + f) \end{array}\right\}.$$

Substituting we have

$$\left.\begin{array}{l} G(A\lambda^2 + b) = -a\lambda\,.\,F \\ F(A\lambda^2 + b) = -a\lambda\,.\,G \end{array}\right\};$$

$$\therefore\ (A\lambda^2 + b)^2 = a^2\lambda^2,$$

or, $A\lambda^2 \pm (2A - C)\,n\,.\,\lambda + (A - C)\,n^2 + gh = 0;$

$$\therefore\ \lambda = \mp\frac{2A - C}{2A}\,.\,n \pm \frac{\sqrt{C^2n^2 - 4A\,.\,gh}}{2A}.$$

In order that the top may not tumble down, these values of λ must be real. Hence we must have

$$n \text{ not less than } \frac{\sqrt{4A \cdot gh}}{C}.$$

143. *Second Method.* When the body is such that two of the principal moments of inertia at the fixed point are equal, the equations of the first method may be changed into another form. Instead of referring the motion of C to two axes Ox', Oy' which move in space in a known manner, we may refer it to two *fixed* axes OX, OY.

Let O be the fixed point; OC the principal axis about which the body rotates, and let OA, OB be two other principal axes which move in the body with an angular velocity $= -n$, and which therefore never deviate far from the fixed axes OX, OY.

The equations of motion referred to the axes OA, OB, OC are by Art. 107,

$$\left.\begin{aligned} A\left(\frac{d\omega_1}{dt} + n\omega_2\right) - (A - C)\, n\omega_2 &= L \\ A\left(\frac{d\omega_2}{dt} - n\omega_1\right) + (A - C)\, n\omega_1 &= M \\ C\,\frac{d\omega_3}{dt} &= N \end{aligned}\right\} \quad \ldots\ldots \quad (1).$$

Let P, Q be the direction-cosines of OC referred to the fixed axes. Then $P = \cos(CX)$, $\therefore\ \dfrac{dP}{dt} = -\sin(CX)\,\dfrac{d(CX)}{dt}$.

But $\omega_2 = -\dfrac{d(CX)}{dt}$ and $CX = \dfrac{\pi}{2}$ nearly.

$$\therefore\ \omega_2 = \frac{dP}{dt},$$

and similarly

$$\omega_1 = -\frac{dQ}{dt}.$$

Substituting these expressions in the equations (1), we get

$$\left.\begin{aligned} A\frac{d^2Q}{dt^2} - Cn\frac{dP}{dt} &= -L \\ A\frac{d^2P}{dt^2} + Cn\frac{dQ}{dt} &= M \end{aligned}\right\}\dots\dots\dots\dots\dots(\text{B}),$$

which are the general equations of small oscillations.

144. Ex. I. *A sphere is suspended from a fixed point by a string and makes small oscillations about the vertical through the point of suspension; find the motion.*

Let l be the length and T the tension of the string; P', Q' the cosines of the angles it makes with two fixed horizontal axes OX, OY drawn through O the point of suspension, and let the positive direction of the axis of z be measured downwards. Let G be the centre of the sphere, and $C'G$ the radius passing through the point of contact C' of the string, and let a be its length. Let $C'G$ produced cut the sphere again in C, and let GC be the positive direction of the axis of C.

Then the equations of motion are, taking the mass as unity,

Dynamical Equations,

$$\frac{d^2x}{dt^2} = -TP',$$

$$\frac{d^2y}{dt^2} = -TQ',$$

$$\frac{d^2z}{dt^2} = g - T,$$

$$\left.\begin{aligned} A\left(\frac{d^2Q}{dt^2} - n\frac{dP}{dt}\right) &= -(yZ - zY) \\ &= Ta(Q' - Q) \\ A\left(\frac{d^2P}{dt^2} + n\frac{dQ}{dt}\right) &= Ta(P' - P) \end{aligned}\right\}.$$

Geometrical Equations,

$$x = lP' + aP,$$

$$y = lQ' + aQ,$$

$$z = l \quad + a.$$

By simplification these reduce to

$$\left.\begin{aligned}
l\frac{d^2P'}{dt^2}+a\frac{d^2P}{dt^2}&=-gP'\\
l\frac{d^2Q'}{dt^2}+a\frac{d^2Q}{dt^2}&=-gQ'\\
\frac{d^2Q}{dt^2}-n\frac{dP}{dt}&=\frac{ag}{A}(Q'-Q)\\
\frac{d^2P}{dt^2}+n\frac{dQ}{dt}&=\frac{ag}{A}(P'-P)
\end{aligned}\right\}.$$

To solve these put

$$P=F\cos(\lambda t+f),\ P'=F'\cos(\lambda t+f),$$
$$Q=G\sin(\lambda t+f),\ Q'=G'\sin(\lambda t+f).$$

Then we have

$$\left.\begin{aligned}
F'(l\lambda^2-g)&=-a\lambda^2F\\
G'(l\lambda^2-g)&=-a\lambda^2G\\
\frac{ag}{A}G'+\left(\lambda^2-\frac{ag}{A}\right)G&=+n\lambda F\\
\frac{ag}{A}F'+\left(\lambda^2-\frac{ag}{A}\right)F&=\quad n\lambda G
\end{aligned}\right\}.$$

Hence we have

$$\left(-\frac{ag}{A}\frac{a\lambda^2}{l\lambda^2-g}+\lambda^2-\frac{ag}{A}\right)^2=(n\lambda)^2,$$

which leads to

$$(l\lambda^2-g)\left(\lambda^2\pm n\lambda-\frac{5g}{2a}\right)=\frac{5}{2}g\lambda^2.$$

This equation gives four *real* values of λ^2. Let the values of λ be $\pm\lambda_1$, $\pm\lambda_2$, $\pm\lambda_3$, $\pm\lambda_4$. Then the oscillation will be represented by $P = F_1 \cos(\lambda_1 t + f_1) + F_2 \cos(\lambda_2 t + f_2)$ + two other terms, with similar expressions for P', Q, Q'.

The equations above will give the values of $\frac{F'}{F}$, $\frac{G}{F}$, $\frac{G'}{F}$ corresponding to each value of λ. Thus we shall have left eight arbitrary constants, viz. F_1, F_2, F_3, F_4, f_1, f_2, f_3, f_4 to be determined by the initial values of

$$P_1, \frac{dP}{dt}, P_1', \frac{dP'}{dt}, \text{\&c.}\dots$$

If we take the negative values of λ, we merely get the same expression over again.

The values of P, P', &c. being known, those of x, y, z may be found, and thus the whole motion may be determined.

145. Ex. II. *A hoop rolls along a perfectly rough horizontal plane, it is required to determine the least angular velocity that it may move with its plane very nearly vertical.*

Supposing the hoop to be originally vertical it will from symmetry roll along a curve which is nearly a straight line. Let this be taken for the axis of x, and let the axis of y be vertical. Let Y, Z be the resolved parts of the friction along the axes of y and z, and R be the normal reaction. Then taking the mass of the hoop as unity we have $R = g$.

The equations of motion of the hoop are

$$A\frac{d^2Q}{dt^2} - 2An\frac{dP}{dt} = -L,$$

$$A\frac{d^2P}{dt^2} + 2An\frac{dQ}{dt} = M.$$

Now, M the moment of the forces about Oy is clearly $= 0$, and $L = -Za - Ra\theta$, where θ is equal to the angle the projection of GC on the plane yz makes with Z. But $\theta = Q$;

$$\therefore -L = Za + agQ.$$

Also if x, y, z be the co-ordinates of G we have

$$\frac{d^2z}{dt^2} = Z, \text{ and } \frac{dz}{dt} = \omega_x y.$$

But $\omega_x = \omega_1 \cos AOx + \omega_2 \cos BOx + \omega_3 \cos COx$

$$= \omega_1 + nP$$

$$= -\frac{dQ}{dt} + nP;$$

$$\therefore -Z = a\frac{d^2Q}{dt^2} - an\frac{dP}{dt}.$$

Substituting in the first equation, we have

$$(A + a^2)\frac{d^2Q}{dt^2} - n(2A + a^2)\frac{dP}{dt} - gaQ = 0,$$

and integrating the second equation

$$\frac{dP}{dt} = -2nQ + \text{constant}.$$

Hence we have

$$\frac{d^2Q}{dt^2} + \frac{2n^2(2A + a^2) - ag}{A + a^2}Q = \text{constant}.$$

Therefore the hoop will roll if

$$n^2 > \frac{ag}{2(2A + a^2)}.$$

If the hoop be a circular arc, $A = \frac{a^2}{2}$, and we have

$$n > \frac{1}{2}\sqrt{\frac{g}{a}}.$$

Let V be the velocity of the centre of the hoop, then we have

$$V > \frac{1}{2}\sqrt{ag}.$$

146. *Third Method.* Suppose the geometrical relations that constrain the body are such that one point of the instantaneous axis is known throughout the motion. Then we know that in small oscillations we may take moments about this point as if it was fixed. This will greatly simplify our equations, for the unknown reactions will generally act at this point.

Through this point, which we will call O, let axes Ox, Oy, Oz, be drawn parallel to the principal axes GA, GB, GC at the centre of gravity, and let $\bar{x}$, $\bar{y}$, h be the co-ordinates of G so that $\bar{x}$, $\bar{y}$, are very small.

Referring to first principles, we have by (101) the general equation

$$\left.\begin{aligned}&\Sigma m\,(y^2+z^2)\,\frac{d\omega_x}{dt}-\Sigma m\,(z^2-y^2)\,\omega_y\,\omega_z+\Sigma m\,yz\,(\omega_z^2-\omega_y^2)\\&\quad-\Sigma m\,xz\left(\frac{d\omega_z}{dt}+\omega_x\,\omega_y\right)-\Sigma m\,xy\left(\frac{d\omega_y}{dt}-\omega_x\,\omega_z\right)\end{aligned}\right\}=L$$

and two other similar equations, where L, M, N are the moments about the axes Ox, Oy, Oz respectively.

In these equations we are to neglect the squares of all small quantities. Hence the squares and products of $\bar{x}$, $\bar{y}$, ω_x, ω_y, $\dfrac{d\omega_z}{dt}$, are to be rejected. Also since the axes are parallel to the principal axes at G we have $\Sigma m\,xz=h\bar{x}$, $\Sigma m\,yz=h\bar{y}$, and $\Sigma m\,xy=\bar{x}\bar{y}$, the mass of the body being taken as unity.

The equations therefore reduce to

$$\left.\begin{aligned}A'\,\frac{d\omega_x}{dt}-(B'-C')\,\omega_y\,n+h\bar{y}\,n^2&=L\\B'\,\frac{d\omega_y}{dt}-(C'-A')\,\omega_x\,n-h\bar{x}\,n^2&=M\\C'\,\frac{d\omega_z}{dt}&=N\end{aligned}\right\},$$

where A', B', C' are the moments of inertia about the axes of Ox, Oy, Oz.

As before let $\left.\begin{aligned} p &= \theta\cos\phi \\ q &= -\theta\sin\phi \end{aligned}\right\}$.

Then the above become, as in Art. 138,

$$\left.\begin{aligned} A'\frac{d^2q}{dt^2} + (A'+B'-C')\,n\frac{dp}{dt} - (B'-C')\,n^2q - h\bar{y}\,n^2 &= -L \\ B'\frac{d^2p}{dt^2} - (A'+B'-C')\,n\frac{dq}{dt} - (A'-C')\,n^2p - h\bar{x}\,n^2 &= M \\ C'\,\frac{d\omega_z}{dt} &= N \end{aligned}\right\}(\mathrm{C}).$$

The moments L, M, N, and the quantities h, $\bar{x}$, $\bar{y}$, must then be found as before in terms of p and q by the geometry of the question, and the equations will then become independent of the position of the instantaneous axis. They will therefore be true throughout the motion.

They may be integrated in the usual manner, and the whole motion may be found.

147. Ex. *An ellipsoid is in rotation about a principal diameter and is placed with the extremity of this diameter in contact with a perfectly rough horizontal plane. Supposing the body to make small oscillations, determine the motion.*

By Art. 138 it is evident that the axes GC, GA, GB make angles with the vertical whose direction-cosines are $1, -p, -q$. Hence the resolved parts of gravity along the axes Oz, Ox, Oy will be $Z=-g$, $X=pg$, $Y=qg$. Let $\bar{x}$, $\bar{y}$, c be the co-ordinates of G referred to these axes, then we have

$$\left.\begin{aligned} L &= -g\bar{y} - cgq \\ M &= g\bar{x} + cgp \\ N &= 0 \end{aligned}\right\}.$$

We must now find $\bar{x}$ and $\bar{y}$ in terms of p and q. Taking the extremity of the axis of C as origin of a new set of axes x', y', z' parallel to GA, GB, GC respectively, the equation to the surface is

$$2z' = \frac{cx'^2}{a^2} + \frac{cy'^2}{b^2},$$

the term $\frac{z'^2}{c}$ being neglected since it is of the order $\left(\frac{x'}{a}\right)^4$.

Let x', y', z' be the co-ordinates of the point O of contact of the ellipsoid with the plane. The equation to the normal at this point

$$\frac{\xi - x'}{\frac{cx'}{a^2}} = \frac{\eta - y'}{\frac{cy'}{b^2}} = \frac{\zeta - z'}{-1}.$$

But this normal is the vertical, hence its direction-cosines are $1, -p, -q$;

$$\therefore\ p = \frac{cx'}{a^2}, \quad q = \frac{cy'}{b^2}.$$

But $x' = -\bar{x}$, and $y' = -\bar{y}$,

$$\therefore\ \bar{x} = -\frac{a^2 p}{c}, \quad \bar{y} = -\frac{b^2 q}{c}.$$

Substituting in the general equations of motion we get

$$\left.\begin{aligned}(6c^2 + b^2)\frac{d^2q}{dt^2} + 12c^2n\frac{dp}{dt} + (b^2 - c^2)\left(6n^2 + \frac{5g}{c}\right)q = 0\\ (6c^2 + a^2)\frac{d^2p}{dt^2} - 12c^2n\frac{dq}{dt} + (a^2 - c^2)\left(6n^2 + \frac{5g}{c}\right)p = 0\end{aligned}\right\} \quad (1).$$

To solve these, let

$$p = F\sin(\lambda t + f),$$
$$q = G\cos(\lambda t + f).$$

Substituting in (1), we shall get a biquadratic equation to determine λ.

In order that the motion may be oscillatory, the four values of λ must be real. Hence by Art. 140 the three quantities

$$\left\{\left(6n^2+5\frac{g}{c}\right)\left(\frac{c^2-a^2}{6c^2+a^2}+\frac{c^2-b^2}{6c^2+b^2}\right)-\frac{144c^4n^2}{(6c^2+a^2)(6c^2+b^2)}\right\}^2$$
$$-4\left(6n^2+5\frac{g}{c}\right)^2\cdot\frac{c^2-a^2}{6c^2+a^2}\cdot\frac{c^2-b^2}{6c^2+b^2},$$
$$(c^2-a^2)(c^2-b^2),$$
$$-\left(6n^2+5\frac{g}{c}\right)\left(\frac{c^2-a^2}{6c^2+a^2}+\frac{c^2-b^2}{6c^2+b^2}\right)+\frac{144c^4n^2}{(6c^2+a^2)(6c^2+b^2)},$$

must all be positive.

If c be the least axis of the ellipsoid, these conditions are satisfied for all values of n. See Art. 140.

If c be the mean axis of the ellipsoid, the second quantity is essentially negative, and no value of n can be found that will make the position of the ellipsoid stable.

If c be the greatest axis of the ellipsoid, the second condition is satisfied. Let

$$H=(c^2-a^2)(6c^2+b^2)+(c^2-b^2)(6c^2+a^2).$$

Then since the first quantity is positive we must have

$$\left\{(144c^4-6H)n^2-5\frac{g}{c}H\right\}^2-4\left(6n^2+\frac{5g}{c}\right)^2(c^2-a^2)(c^2-b^2)$$
$$=\text{positive quantity} \;\ldots\ldots\ldots\ldots\ldots\; (A),$$

and since the third expression must be positive we have

$$n^2>\frac{5\frac{g}{c}H}{144c^4-6H}\;\ldots\ldots\ldots\ldots\ldots\ldots\; (C).$$

If this value of n^2 be substituted in the expression (A) it makes (A) negative, hence this value of n^2 lies between the roots of the quadratic equation formed by equating (A) to zero. In order therefore that the three conditions may be satisfied we must have n^2 greater than the greatest root of the equation formed by equating (A) to zero. It is evident that one of the roots of this equation is essentially positive.

148. *Fourth Method.* When the body performing small oscillations is a rod, we may obtain the linear equations of motion very simply by referring to the original equations of motion in Chap. II. as in the following problem.

A uniform heavy rod suspended from a fixed point O by a string, makes small oscillations about the vertical. Determine the motion.

Let O be taken as origin, and let the axis of z be measured vertically downwards; let l be the length of the string, $2a$ the length of the rod. Let p', q'; p, q be the cosines of the angles the string and rod respectively make with the axes of x and y, and let u be the distance of any element du of the rod from that extremity to which the string is attached. Then the co-ordinates of the element will be

$$\left.\begin{aligned} x &= lp' + up \\ y &= lq' + uq \\ z &= l + u \end{aligned}\right\} \text{..................} \quad (1).$$

The equations of motion of the centre of gravity will be

$$\left.\begin{aligned} l\frac{d^2p'}{dt^2} + a\frac{d^2p}{dt^2} &= -\frac{Tp'}{M} \\ l\frac{d^2q'}{dt^2} + a\frac{d^2q}{dt^2} &= -\frac{Tq'}{M} \\ 0 &= g - \frac{T}{M} \end{aligned}\right\} \text{..................} \quad (2),$$

where T is the tension of the string, and M the mass of the rod. By D'Alembert's Principle, the equation of moments round x will be

$$\begin{aligned} \Sigma du\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right) &= \Sigma du\,(yZ - zY) \\ &= \Sigma du\,(yg), \end{aligned}$$

which becomes by (1)

$$\int_0^{2a} du\left\{-(l+u)\left(l\frac{d^2q'}{dt^2} + u\frac{d^2q}{dt^2}\right)\right\} = 2ag\,(lq' + aq),$$

or

$$-2al\left(l\frac{d^2q'}{dt^2}+a\frac{d^2q}{dt^2}\right)-2la^2\frac{d^2q'}{dt^2}-\frac{8a^3}{3}\frac{d^2q}{dt^2}=2ag\,(lq'+aq),$$

which by equation (2) reduces to

$$l\frac{d^2q'}{dt^2}+\frac{4}{3}a\frac{d^2q}{dt^2}=-gq.$$

Therefore the four equations of motion are

$$\left.\begin{aligned} l\frac{d^2p'}{dt^2}+a\frac{d^2p}{dt^2}&=-gp' \\ l\frac{d^2p'}{dt^2}+\frac{4}{3}a\frac{d^2p}{dt^2}&=-gp \end{aligned}\right\}\dots\dots\dots\dots\dots (3),$$

and two similar equations for q, q'.

To solve these, put

$$p'=F\sin(\lambda t+\alpha),\quad p=G\sin(\lambda t+\alpha),$$

we get

$$l\lambda^2F+a\lambda^2G=gF,$$

$$l\lambda^2F+\frac{4}{3}a\lambda^2G=gG;$$

$$\therefore\ \lambda^4-\frac{4a+3l}{al}g\,.\,\lambda^2+\frac{3g^2}{al}=0,$$

and the values of λ may be found from this equation.

SECT. V. *On steady Motion and small Oscillations.*

149. Hitherto we have supposed the body to be performing small oscillations about its position of equilibrium. But if the body have a steady motion in space and perform small oscillations about this moving position, the equations become more complicated. To simplify the problem, let two of the principal moments A, B, at the fixed point be equal.

The two following problems will be sufficient to show the mode of proceeding in the cases *first*, when one point of the body is absolutely fixed; and *secondly*, when the body rolls freely on a rough plane.

150. PROB. I. *A body two of whose principal moments at the centre of gravity G are equal turns about a fixed point O, in the axis of unequal moment under the action of gravity. The axis GO being inclined to the vertical at an angle α, and revolving about it with a uniform angular velocity, find the condition that the motion may be steady and the time of a small oscillation.*

This applies to the case of a top spinning with its vertex O on a perfectly rough plane.

Let the fixed point O be taken as the origin, and let the axis of z be vertical. Let the line OG drawn from O through G the centre of gravity of the body be called the axis of the body. Let the moving axes OA, OB, OC be so chosen that OC is the axis of the body, and that Oz, OC, OA are always in one plane. Then by Art. 109 the equations of motion are

$$\left.\begin{aligned} A\left(\frac{d\omega_1}{dt} - \omega_2\frac{d\chi}{dt}\right) - \left(A - C\right)\omega_2\omega_3 &= 0 \\ A\left(\frac{d\omega_2}{dt} + \omega_1\frac{d\chi}{dt}\right) + \left(A - C\right)\omega_3\omega_1 &= gh\sin\theta \\ C\frac{d\omega_3}{dt} &= 0 \end{aligned}\right\} \quad \ldots\ldots(1),$$

and the geometrical equations are

$$\left.\begin{aligned} \frac{d\theta}{dt} &= \omega_2 \\ \frac{d\psi}{dt}\sin\theta &= -\omega_1 \\ -\frac{d\chi}{dt} + \frac{d\psi}{dt}\cos\theta &= \omega_3 \end{aligned}\right\} \quad \ldots\ldots(2).$$

Eliminating ω_1 ω_2, and putting $\omega_3 = n$, we get

$$A\sin\theta\frac{d^2\psi}{dt^2} + 2A\cos\theta\frac{d\psi}{dt}\frac{d\theta}{dt} - Cn\frac{d\theta}{dt} = 0 \text{..........} (3),$$

$$A\frac{d^2\theta}{dt^2}\ A\cos\theta\sin\theta.\left(\frac{d\psi}{dt}\right)^2 + Cn\sin\theta\frac{d\psi}{dt} = gh\sin\theta \text{ ... } (4).$$

To find the steady motion.

When the motion is steady both θ and $\frac{d\psi}{dt}$ are constants.

Let $\theta = \alpha$, $\frac{d\psi}{dt} = \lambda$, then the equation (3) is satisfied and (4) becomes

$$-A\cos\alpha\sin\alpha\,\lambda^2 + Cn\sin\alpha\,\lambda = gh\sin\alpha.$$

Rejecting the factor $\sin\alpha = 0$ because α is *not* small, we have

$$n = \frac{A\cos\alpha\,\lambda^2 + gh}{C\lambda} \text{ .... } (5).$$

This is the relation which must hold between the angular velocity of the body about its axis, and the angular velocity of that axis about the vertical, when the motion is *steady*. But this relation is not necessary if the axis of the body be vertical.

Solving this equation, we have

$$\lambda = \frac{Cn \pm \sqrt{C^2n^2 - 4ghA\cos\alpha}}{2A\cos\alpha},$$

hence in order that the motion may be steady, we must have

$$n^2 \text{ not less than } \frac{4ghA\cos\alpha}{C^2}.$$

When α and n are given, we can make the body move with either of these values of λ, by giving ω_1 its proper initial value determined by the equation

$$\omega_1 = -\lambda\sin\alpha.$$

To find the small oscillation.

Let $\theta = \alpha + \theta'$, and $\frac{d\psi}{dt} = \lambda + \frac{d\psi'}{dt}$, where θ' and $\frac{d\psi'}{dt}$ are small quantities whose squares are to be neglected. Substituting these in (3) and (4), and writing for Cn its value obtained from (5), we have

$$\left.\begin{aligned} A\lambda \sin\alpha \frac{d^2\psi'}{dt^2} - (gh - A\lambda^2\cos\alpha)\frac{d\theta'}{dt} = 0 \\ A\lambda \frac{d^2\theta'}{dt^2} + \sin\alpha\,(gh - A\lambda^2\cos\alpha)\frac{d\psi'}{dl} + \lambda^3 A \sin^2\alpha\theta' = 0 \end{aligned}\right\}.$$

To solve these, put

$$\theta' = F\sin(pt+f), \text{ and } \psi' = G\cos(pt+f).$$

Substituting, we have

$$\left.\begin{aligned} A\lambda\sin\alpha\,.\,p^2G = -(gh - A\lambda^2\cos\alpha)\,Fp \\ (A\lambda p^2 - \lambda^3 A\sin^2\alpha)\,F = -(gh - A\lambda^2\cos\alpha)\sin\alpha\,.\,Gp \end{aligned}\right\}.$$

Hence multiplying these equations together, we have

$$p^2 = \frac{A^2\lambda^4 - 2ghA\cos\alpha\,.\,\lambda^2 + g^2h^2}{A^2\lambda^2},$$

and the required time is $\frac{2\pi}{p}$*. It is evident that p^2 is always positive, and therefore both the values of λ given by (5) correspond to stable motions.

To find the friction at the fixed point.

The equations for the motion of the centre of gravity are by Art. 63,

$$\left.\begin{aligned} \frac{du}{dt} - v\frac{d\psi}{dt} = X \\ \frac{dv}{dt} + u\frac{d\psi}{dt} = Y \end{aligned}\right\},$$

where u and v are the velocities of the centre of gravity resolved along and perpendicular to the projection of OA on the horizontal plane, and X, Y are the resolved parts of the frictions in the same directions.

* This expression was given by the Rev. N. M. Ferrers, of Gonville and Caius College, as the result of a problem proposed by him for solution in the Mathematical Tripos, 1859.

Since the point O is fixed, we have

$$\left.\begin{aligned} u &= \omega_2 h \cos\theta \\ v &= -\omega_1 h \cos\theta \end{aligned}\right\};$$

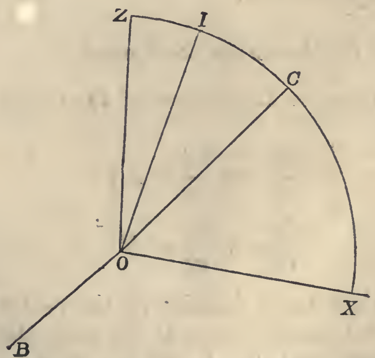
ω_1 and ω_2 are known from equations (2), and thus X and Y may be found.

When the motion is steady, we have from (2) $\omega_2 = 0$, and $\omega_1 = -\lambda \sin\alpha$; $\therefore\ u = 0$ and $v = \lambda \sin\alpha \cos\alpha h$;

$$\left.\begin{aligned} \therefore\ X &= -\lambda^2 \sin\alpha \cos\alpha h \\ Y &= 0 \end{aligned}\right\},$$

thus the whole friction acts in the vertical plane ZCA. Since G describes a horizontal circle, the force acting on it must tend to the centre, and therefore this result might have been anticipated.

151. We may also determine the *steady motion* very simply by another process. Let OC be the axis of the body, OI the instantaneous axis of rotation, OZ the vertical. Then when the motion is steady, these three must be in one vertical



plane which revolves about OZ with a uniform angular velocity n. Let Ω be the angular velocity about OI, then $\Omega \cos IC = n$. Let OB be the horizontal axis about which gravity tends to turn the body, then OB is perpendicular to the plane ZOC.

Since gravity generates an angular velocity $\frac{gh \sin \alpha}{A} dt$ in the time dt about OB, therefore by the parallelogram of angular velocities, the instantaneous axis OI has moved in the time dt through an angle $\frac{gh \sin \alpha}{A\Omega} dt$ in a plane perpendicular to the plane ZOI. Hence the angular velocity of I round Z due to the action of the forces is

$$\frac{d\psi_1}{dt} = \frac{gh \sin \alpha}{A\Omega} \cdot \frac{1}{\sin IZ}.$$

Also by Art. 136 the angular velocity of I round C due to the inertia of the body is $\frac{A-C}{A} n$, hence the angular velocity of I round Z is

$$\frac{d\psi_2}{dt} = \frac{A-C}{A} n \cdot \frac{\sin IC}{\sin IZ},$$

and the whole angular velocity is the sum of these two, i. e.

$$\lambda = \left(\frac{gh \sin \alpha}{An} \cot IC + \frac{A-C}{A} n\right) \frac{\sin IC}{\sin IZ}$$

$$= \frac{gh \cdot \sin \alpha \cdot \cot IC + (A-C) n^2}{An (\sin \alpha \cdot \cot IC - \cos \alpha)}.$$

But when the motion is steady OZ, OI and OC are all in one plane. Now the angular velocity of C round I is Ω, and therefore its angular velocity round Z is

$$\lambda = \Omega \frac{\sin IC}{\sin ZC} = \frac{n}{\cos IC} \cdot \frac{\sin IC}{\sin \alpha}.$$

Hence, $\tan IC = \frac{\lambda \sin \alpha}{n}.$

Substituting this value of $\tan IC$ in the first value of λ, we get

$$\frac{gh}{\lambda} = Cn - A\lambda \cos \alpha,$$

the same expression as before.

152. The motion of a body about a fixed point under the action of any forces may in many cases be constructed by making the momental ellipsoid roll on some surface to be determined by the conditions of the problem. But this representation of the motion is not always a convenient one. In the case just considered, the steady motion may be represented by making the ellipsoid roll on the surface of a right cone whose axis is vertical.

The ellipsoid is in this case a spheroid whose axis is OC. Let OC be taken as the axis of z, and let Ox be drawn to the left of OC. Then the equation to the section of the ellipsoid by the plane COZ is

$$Ax^2 + Cz^2 = \epsilon^4 \text{.......................} (1).$$

Let x, z be the co-ordinates of the point in which the instantaneous axis OI cuts the ellipsoid, and let r be the corresponding radius vector, and let the angle $IOC = \beta$. Then

$$\left.\begin{aligned} A\sin^2\beta + C\cos^2\beta &= \frac{\epsilon^4}{r^2} \\ \tan\beta &= \frac{\lambda\sin\alpha}{n} \end{aligned}\right\} \text{.................} (2).$$

The equation to the tangent plane at the extremity of the radius vector r is

$$A\sin\beta \,.\, x + C\cos\beta \,.\, z = \frac{\epsilon^4}{r} \text{ ....} (3),$$

and the equation to OZ, (figure, Art. 151) is

$$x = z\tan\alpha \text{ ....} (4).$$

Let γ be the semi-angle of the cone, then γ is the inclination of OZ to the tangent plane (3);

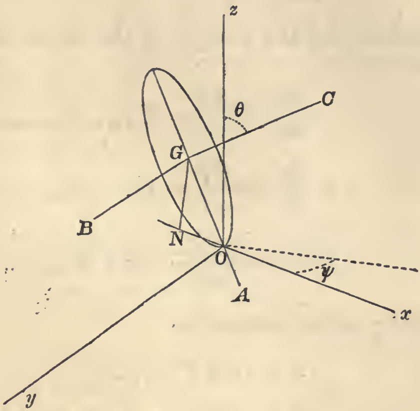
$$\therefore \tan\gamma = \frac{\dfrac{C}{A}\cot\beta + \tan\alpha}{1 - \dfrac{C}{A}\cot\beta \,.\, \tan\alpha},$$

after substituting for $\tan\beta$, we get

$$\cot\gamma = -\frac{gh\sin\alpha}{A\lambda^2 + gh\cos\alpha}.$$

153. PROB. 2. *An infinitely thin circular disc moves on a perfectly rough horizontal plane in such a manner as to preserve a constant inclination* α *to the horizon. Find the condition that the motion may be steady and the time of a small oscillation.*

Let the mass of the disc be taken as unity, let a be its radius. Let the axis of z be vertical, and let the axes GA, GB, GC, drawn through G, the centre of gravity, move in the body so that GC is normal to the plane of the disc, and that the three axes Gz, GC, GA are all in one vertical plane.



Let O be the point of the disc in contact with the plane; X, Y the frictions at O resolved parallel and perpendicular to the plane CGA. Let R be the reaction at O normal to the horizontal plane. Let u, v be the horizontal velocities of the centre of gravity resolved along and perpendicular to the plane CGA. Let the rest of the notation be the same as before.

The equations of motion about the centre of gravity will be

$$A\left(\frac{d\omega_1}{dt} - \omega_2\frac{d\chi}{dt}\right) - (A - C)\,\omega_2\omega_3 = 0 \quad\ldots\ldots(1),$$

$$A\left(\frac{d\omega_2}{dt} + \omega_1\frac{d\chi}{dt}\right) + (A - C)\,\omega_3\omega_1 = -Xa\sin\theta - Ra\cos\theta \ldots(2),$$

$$C\frac{d\omega_3}{dt}=aY \quad\text{......................}\quad (3),$$

and the geometrical equations by the second part of Art. 108, are

$$\frac{d\theta}{dt}=\omega_2 \quad\text{......................}\quad (4),$$

$$\frac{d\psi}{dt}\sin\theta=-\omega_1 \quad\text{..................}\quad (5),$$

$$-\frac{d\chi}{dt}+\frac{d\psi}{dt}\cos\theta=\omega_3 \quad\text{......................}\quad (6).$$

The equations for the motion of the centre of gravity are by Art. 63,

$$\frac{du}{dt}-v\frac{d\psi}{dt}=X \quad\text{........................}\quad (7),$$

$$\frac{dv}{dt}+u\frac{d\psi}{dt}=Y \quad\text{........................}\quad (8),$$

$$\frac{d^2z}{dt^2}=-g+R \quad\text{................}\quad (9),$$

and the geometrical equations are

$$u=a\sin\theta\,.\,\omega_2 \quad\text{........................}\quad (10),$$

$$v=-a\omega_3 \quad\text{..............................}\quad (11),$$

$$z=a\sin\theta \quad\text{............................}\quad (12).$$

To solve these, we must eliminate u, v, and z. Then since the square of $\frac{d\theta}{dt}$ may be neglected, we have

$$\left.\begin{aligned} X&=a\sin\alpha\frac{d^2\theta}{dt^2}+a\omega_3\frac{d\psi}{dt}\\ Y&=-a\frac{d\omega_3}{dt}+a\omega_2\sin\theta\frac{d\psi}{dt}\\ R&=g+a\cos\theta\frac{d^2\theta}{dt^2}\end{aligned}\right\}\quad\text{.............}\quad (\mathrm{I}).$$

Substituting these in (1), (2), (3) and remembering (5) we have

$$\left.\begin{aligned}
&A'\left(\frac{d\omega_1}{dt}-\omega_2\frac{d\chi}{dt}\right)-(B'-C')\,\omega_2\omega_3=0\\
&B'\frac{d\omega_2}{dt}+A\omega_1\frac{d\chi}{dt}+(A'-C')\,\omega_3\omega_1=-ga\cos\theta\\
&\qquad\quad C'\frac{d\omega_3}{dt}-(A'-B')\,\omega_1\omega_2=0
\end{aligned}\right\}\ldots\ldots\text{(II)},$$

where $\qquad A'=A,\ \ B'=B+a^2,\ \ C'=C+a^2.$

These equations might have been obtained by taking O as the origin of moments. To do this we must apply to every element of the body an acceleration equal and opposite to that of O. The acceleration of O due to the steady motion is $a\omega_1\dfrac{d\chi}{dt}$ perpendicular to the plane of the disc. In taking moments this must be supposed to act at G the centre of gravity of the disc by Art. 65. The acceleration of O due to the small oscillation may be neglected.

To find the steady motion.

We have $\omega_1,\ \omega_2,\ \omega_3,\ \dfrac{d\psi}{dt},\ \theta,$ all constants. Let $\omega_3=n$, $\dfrac{d\psi}{dt}=\lambda,\ \theta=\alpha$, then substituting in (4), (5), (6), and (II) we have

$$\omega_2=0,$$

$$\omega_1=-\lambda\sin\alpha,$$

$$(2A+a^2)\,\omega_3=A\lambda\cos\alpha-\frac{ga}{\lambda}\cot\alpha,$$

since $C=2A$.

This relation must exist between the angular velocity of the disc about its axis and the angular velocity of the horizontal tangent, in order that the motion may be steady. The point of the disc in contact with the plane describes a circle on the plane. Let r be the radius of this circle. Then since the disc turns round once in the time $\frac{2\pi}{\omega_3}$;

$$\therefore -r\,.\,\frac{2\pi}{\omega_3}\lambda = 2\pi a; \quad \therefore r = -\frac{\omega_3}{\lambda}a;$$

$$\therefore (2A+a^2)\,r = -Aa\,.\cos\alpha + \frac{ga^2}{\lambda^2}\cot\alpha.$$

To find the small oscillation.

Let $\theta = \alpha + \theta'$, $\frac{d\psi}{dt} = \lambda + \frac{d\psi'}{dt}$, $\omega_3 = n + \omega_3'$ where θ', $\frac{d\psi'}{dt}$, ω_3' are small quantities whose squares and higher powers are to be neglected. Substituting for ω_1, ω_2, $\frac{d\chi}{dt}$, from (4), (5), (6), the first of equations (II) becomes

$$A\sin\alpha\,\frac{d^2\psi'}{dt^2} + 2A\,(\cos\alpha\,.\,\lambda - n)\,\frac{d\theta'}{dt} = 0 \text{ .... (13).}$$

The third becomes

$$(2A+a^2)\,\frac{d\omega_3}{dt} - a^2\sin\alpha\,\lambda\,\frac{d\theta}{dt} = 0,$$

$$\therefore (2A+a^2)\,(\omega_3 - n) = a^2\sin\alpha\lambda\,(\theta-\alpha) \text{ ....(14).}$$

The second becomes

$$(A+a^2)\,\frac{d^2\theta}{dt^2} + (2A+a^2)\sin\theta\omega_3\,\frac{d\psi}{dt}$$
$$- A\sin\theta\cos\theta\left(\frac{d\psi}{dt}\right)^2 = -ga\cos\theta,$$

which reduces to

$$(A+a^2)\,\frac{d^2\theta'}{dt^2} + L\theta' - M\sin\alpha\,\frac{d\psi'}{dt} = 0 \text{ .... (15),}$$

where

$$L = -A\lambda^2 \cos 2\alpha + (2A + a^2)\, n\lambda \cos \alpha - ga \sin \alpha,$$

$$M = 2A \cos \alpha \,.\, \lambda - (2A + a^2)\, n.$$

Integrating (13) and substituting in (15), we have

$$(A + a^2)\frac{d^2\theta'}{dt^2} + \{L + 2M(\cos\alpha \,.\, \lambda - n)\}\,\theta' = 0, \quad \text{...(16)}.$$

The constant introduced by integrating (13) would give rise to a constant term in the value of θ' obtained by integrating (16). But all the constant part of θ has been supposed to be included in its mean value $\theta = \alpha$. Hence this constant mest be omitted.

Therefore if T be the time of a small oscillation

$$\left(\frac{2\pi}{T}\right)^2 (A + a^2) = (1 + 2\cos^2\alpha)\, A\lambda^2 - 2n\lambda \cos\alpha\, (3A + a^2),$$
$$+ 2n^2 (2A + a^2) - ga \sin \alpha.$$

The frictions at the point O may be found by equations (7) to (12).

EXAMPLES.

SECTION I.

1. If ω_x, ω_y, ω_z be the angular velocities about the coordinate axes, find the locus of those particles whose velocity is $a\omega_x$.

2. The locus of points in a body moving about a fixed point, which at any proposed instant have the same velocity, is a circular cylinder.

3. A body has an angular velocity ω about the axis

$$\frac{x - a}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n},$$

where $l^2+m^2+n^2=1$. The motion is equivalent to rotations $l\omega$, $m\omega$, $n\omega$ about the co-ordinate axes, and translations

$$(m\gamma-n\beta)\,\omega,\ \ (n\alpha-l\gamma)\,\omega,\ \ (l\beta-m\alpha)\,\omega,$$

in direction of them.

4. A circular disc revolves with a uniform angular velocity ω about an axis through its centre perpendicular to its plane, while its centre describes a circle of radius a with uniform angular velocity Ω about a point in the plane of the disc. Prove that the motion at any instant is exhibited by a single rotation, and the locus of the instantaneous axis is a cylinder of radius $\dfrac{a\omega}{\omega+\Omega}$. Explain the result when ω, Ω are equal and contrary in direction.

5. A body has equal angular velocities about two axes which neither meet nor are parallel. Prove that the central axis of the motion is equally inclined to each of the axes.

6. If, in a rigid body moving in any manner about a fixed point, a series of points be taken along any straight line in the body, and through these points straight lines be drawn in the direction of the instantaneous motion of the points, prove that the locus of these straight lines is an hyperbolic paraboloid.

7. In the motion of a geometrical figure there exist angular velocities inversely proportional to $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$ round three lines forming three edges of a cube which do not meet, symmetrically chosen with respect to the axes of co-ordinates drawn parallel to them through the centre of the cube. Prove that the figure rotates about the line

$$(\beta-\gamma)\,x+(\gamma-\alpha)\,a=(\gamma-\alpha)\,y-(\beta-\gamma)\,a=(\alpha-\beta)\,z,$$

where $2a=$ the edge of the cube.

SECTION II.

8. A body turning about a fixed point of it is acted on by forces which always tend to produce rotation about an axis at right angles to the instantaneous axis; show that the angular velocity cannot be uniform unless

$$\frac{C-B}{A}+\frac{B-A}{C}+\frac{A-C}{B}=0,$$

A, B, C being the principal moments of inertia with respect to the fixed point.

9. If forces act on a homogeneous spheroid tending always to produce rotation about an axis (α) in the plane of the equator, the instantaneous axis will describe a circular cone in the body about its polar axis; but the angular velocity about the instantaneous axis will not be uniform unless the axis α be always at right angles to the instantaneous axis.

10. If two of the principal moments of inertia be equal and the body begin to rotate about an axis perpendicular to that of unequal moment, under the action of a couple varying as the cosecant of the angle which the instantaneous axis makes with the axis of unequal moment and in a plane perpendicular to that axis, determine the position of the instantaneous axis in the body in terms of the time.

11. If a rough plane inclined at an angle α to the horizon be made to revolve with uniform angular velocity ω, about a normal Oz and a sphere be placed upon it, show that the path of its centre will be a prolate, a common, or a curtate cycloid according as the point at which the sphere is initially placed is without, upon, or within the circle, whose equation is

$$x^2+y^2=\frac{35g\sin\alpha}{2\omega^2}x,$$

the axis Oy being horizontal

Show also that the path will be a horizontal straight line, if the sphere be initially placed at the centre of the circle.

12. A right cone rolls upon the inner surface of another right cone, having the same vertex, with a uniform angular velocity ω, find the couple impressed upon the rolling cone necessary to produce such motion.

13. A hollow cone, the internal surface of which is perfectly rough, is fixed in a position in which its axis is inclined at an angle (α) to the vertical, and a solid cone of smaller vertical angle is placed inside, its vertex coinciding with the vertex of the fixed cone, and allowed to oscillate: show that the length of the simple isochronous pendulum is

$$\frac{4k^2}{3h}\,\frac{\sin(\beta-\gamma)}{\sin\alpha\sin^2\gamma},$$

2β and 2γ being the vertical angles, h the height of the moving cone, and k its radius of gyration about a generating line.

14. A segment of a solid of revolution bounded by a plane perpendicular to its axis, rests with its plane surface in contact with a perfectly rough horizontal plane, which is revolving with a varying angular velocity ω, about a vertical axis, show that if $h =$ height of the centre of gravity of the solid above the plane, c the distance of its axis from the axis of rotation, and r the radius of the base, the solid will upset, if

$$hc\sqrt{\omega^4+\left(\frac{d\omega}{dt}\right)^2} > gr.$$

15. A segment of a surface of revolution cut off by a plane perpendicular to the axis is placed with its curved surface in contact with a smooth horizontal plane. Supposing a rotation n to be communicated to it about its axis of figure when inclined at an angle α to the vertical, prove that

$$\frac{Cn}{A} \cdot \left(\frac{\cos\alpha - \cos\theta}{\sin\theta}\right)^2 + \left(\frac{d\theta}{dt}\right)^2 = 2g\,(h - z),$$

where θ is the inclination of the axis of figure to the vertical; C is the moment of inertia about the axis; z is the altitude of the centre of gravity at the time t and is a known function of θ, and h is the initial value of z.

16. A hoop $AGBF$ revolves about AB its diameter as a fixed vertical axis. GF is a horizontal diameter of the same circle which is without mass and which is rigidly connected to the circle; DC is another hoop having GF passing through its centre and which can turn freely about GF. If Ω, Ω', ω, ω', be the greatest and least angular velocities about AB, GF respectively, prove that

$$\Omega \,.\, \Omega' = \omega^2 - \omega'^2.$$

17. A circular disc has a rod CO rigidly fixed to it, passing through its centre C perpendicular to its plane. The point O is fastened to a point about which CO can freely turn. The whole system is under the action of gravity. Supposing the initial position of CO to be very nearly horizontal and the disc to be set in rotation about CO, it is required to determine the motion.

18. If a homogeneous sphere roll on a perfectly rough plane under the action of any forces whatever, of which the resultant passes through the centre of the sphere, the motion of the centre of gravity is the same as if the plane were smooth and all the forces were reduced in a certain constant ratio.

Prove also that the plane is the only surface which possesses this property.

19. Prove that a billiard ball, being considered as a sphere partly rolling and partly sliding on a horizontal plane, describes a parabola, except in the particular case when it moves in a right line.

20. A sphere rolls along the inner surface of a fixed circular cylinder with its axis horizontal, it is required to determine the motion of the sphere, the surface of the cylinder being perfectly rough, and the motion not in a plane perpendicular to the axis of the cylinder. Also find the least velocity at the lowest point which will make the sphere go completely round, without leaving the surface of the cylinder.

21. A uniform sphere is placed in contact with the exterior surface of a perfectly rough cone. Its centre is acted on by a force, the direction of which always meets the axis of the cone at right angles, and the intensity of which varies inversely as the cube of the distance from that axis. Prove that, if the sphere be properly started, its path upon the cone will meet each generating line in the same angle.

Section III.

22. If a right circular cone whose altitude a is double the radius of its base turn about its centre of gravity as a fixed point, and be originally set in motion about an axis inclined at an angle α to the axis of figure, the vertex of the cone will describe a circle whose radius is $\frac{3}{4} a \sin \alpha$.

23. A circular plate revolves about its centre of gravity at a fixed point. If an angular velocity ω were originally impressed on it about an axis making an angle α with its plane, a normal to the plane of the disc will make a revolution in space in time $\dfrac{2\pi}{\omega\sqrt{1+3\sin^2\alpha}}$.

24. If a body be turning about a fixed point under the action of no forces, and if

$$\left(\frac{1}{C}-\frac{1}{B}\right)\tan^2\phi = \frac{1}{A}-\frac{1}{C}$$

at any instant when A is finite, then ϕ will be invariable and ψ will increase uniformly. Find the values of ω_1, ω_2, ω_3 in this case.

25. If a solid of revolution be moving about its centre of gravity fixed, show that the plane containing the axis of figure and the instantaneous axis revolves uniformly about a line in itself and that the axes cannot be equally inclined to this line unless $C > 2A$, and in that case the inclination equals $\frac{1}{2}\cos^{-1}\frac{A}{C-A}$, C being the moment of inertia about the axis of figure, and A that about a line perpendicular to it.

26. If ω_1, ω_2, ω_3 be the angular velocities about the principal axes at any fixed point, prove that when all the impressed forces pass through the fixed point

$$\frac{A(\omega_1^2 - h_1^2)}{B-C} = \frac{B(\omega_2^2 - h_2^2)}{C-A} = \frac{C(\omega_3^2 - h_3^2)}{A-B}.$$

27. A right cone the base of which is an ellipse is supported at G the centre of gravity, and has a motion communicated to it about an axis through G perpendicular to the line joining G, and the extremity B of the axis minor of the base, and in the plane through B and the axis of the cone. Determine the position of the invariable plane.

Result. The normal to the invariable plane lies in the plane passing through the axis of the cone and the axis of instantaneous rotation, and makes an angle $\tan^{-1}\frac{b}{h}\cdot\frac{h^2+4a^2}{a^2+b^2}$.

28. A body is revolving about its principal axis of mean moment, and an additional angular velocity about its axis of greatest or least moment is impressed upon it: show that the instantaneous axis will describe within the body a cone of which the internal axis is the axis about which the additional angular velocity is impressed.

29. A lamina of any form rotating with an angular

velocity ω about an axis through its centre of gravity perpendicular to its plane has an angular velocity

$$\left(\frac{A+B}{A-B}\right)^{\frac{1}{2}}\omega$$

impressed upon it about its principal axis of least moment, A and B being its moments of inertia about the principal axes of mean and least moment: show that its angular velocities about the principal axes at any time t are

$$\frac{2\omega}{\epsilon^{\omega t}+\epsilon^{-\omega t}} - \left(\frac{A+B}{A-B}\right)^{\frac{1}{2}}\omega\,\frac{\epsilon^{\omega t}-\epsilon^{-\omega t}}{\epsilon^{\omega t}+\epsilon^{-\omega t}},$$

$$\text{and } \left(\frac{A+B}{A-B}\right)^{\frac{1}{2}}\frac{2\omega}{\epsilon^{\omega t}+\epsilon^{-\omega t}},$$

and that it will ultimately revolve about its axis of mean moment.

30. A rigid body not acted on by any force is in motion about its centre of gravity: prove that if the instantaneous axis be at any moment situated in the plane of contact of either of the right circular cylinders described about the central ellipsoid, it will be so throughout the motion.

If a, b, c be the semi-axes of the central ellipsoid, arranged in descending order of magnitude, e_1, e_2, e_3 the eccentricities of its principal sections, Ω_1, Ω_2, Ω_3 the initial component angular velocities of the body about its principal axes, prove that the condition that the instantaneous axis should be situated in the plane above described is

$$\frac{\Omega_1}{e_1} = \frac{ab}{c^2}\frac{\Omega_3}{e_3}.$$

Prove also that if the preceding condition be satisfied, the position of the instantaneous axis with respect to the principal axes of the body, at any time t, will be determined by the equations

$$x = \frac{2ay}{be_2(\epsilon^{lt} - \epsilon^{-lt})} = \frac{a-b}{c^2 e_3},$$

where $\frac{1}{h^2} = \frac{\Omega^2_1}{a^2} + \frac{\Omega^2_2}{b^2} + \frac{\Omega^3_2}{c^2}$ and $l = \frac{e_3 e_1 a}{h}$.

31. A body is moving about a fixed point under the action of no forces, and a small particle, mass m, is tied to a given point on its surface by a very short string, and moves with the body; determine the tension of the string, the mass of the particle being neglected in comparison of that of the body.

32. If a body move about a fixed point under the action of no forces, and if the instantaneous axis describe the separating polhode, prove that there is a line fixed in the body which always lies in a certain plane fixed in space.

33. The "measure of curvature" of an ellipsoid along any polhode is constant.

Sections IV. and V.

34. A uniform solid of revolution, of which the mass is M, and the principal moments of inertia about its axis of figure, and about any other principal axis through its centre of gravity, are respectively C and A, floats in unstable equilibrium, with its axis vertical, in a fluid of specific gravity greater than its own, the depth of the metacentre below the centre of gravity being c. Prove that the relative equilibrium of the solid will be rendered stable if it be made to rotate uniformly about its axis, with an angular velocity greater than

$$\frac{2}{c}\sqrt{(A + Mc^2)\,Mgc}.$$

35. A perfectly rough sphere is placed on another perfectly rough fixed sphere near the highest point. And the upper sphere has an angular rotation ω about its diameter

through the point of contact; prove that its equilibrium will be stable if $\omega^2 > \dfrac{35g\,(a+b)}{a^2}$ where a is the radius of the fixed sphere, and b the radius of the moving sphere.

36. An ellipsoid is placed with one of its vertices in contact with a smooth horizontal plane. What angular velocity of rotation must it have about the vertical axis in order that the equilibrium may be stable?

Result. Let a, b, c be the semi-axes, c the vertical axis, then the angular velocity must be greater than

$$\sqrt{\frac{5g}{c} \cdot \frac{\sqrt{c^4 - a^4} + \sqrt{c^4 - b^4}}{a^2 + b^2}}\,.$$

37. A uniform rod, moveable about one extremity, moves in such a manner as to make always nearly the same angle α with the vertical; show that the time of a small oscillation is

$$\pi \sqrt{\frac{2a}{3g} \cdot \frac{\cos\alpha}{1 + 3\cos^2\alpha}}\,,$$

a being the length of the rod.

38. A hemisphere of radius a is placed on a perfectly rough plane with its base inclined at an angle α to the horizon, and is set in rotation about its axis of figure with an angular velocity n. Prove that if the motion be steady, the axis of figure will revolve round the vertical with an angular velocity λ given by

$$\lambda^2 \{A\cos\alpha + (h\cos\alpha - a)\,(h - a\cos\alpha)\} + Cn\lambda = -gh,$$

except when $\alpha = 0$: where h is the distance of the centre of gravity from the centre, and C is the moment of inertia about the axis of figure. Prove also that the radius of the circle described on the horizontal plane by the point of contact is $a\sin\alpha \,.\, \cos\alpha$. Find also the time of a small oscillation.

39. A rigid body is attached to a fixed point by a weightless string, length l, which is connected with the body by a socket (permitting the body to rotate freely without

twisting the string) at a point on its surface where an axis through its centre of gravity, about which the radius of gyration is a maximum or minimum $= k$, meets it. The body is set rotating with an angular velocity ω about such axis placed vertically; the string, which is tight, forms an angle α with the vertical, and is then let go; show that it will ultimately revolve with the uniform angular velocity

$$\sqrt{\omega^2 + \frac{2lg \sin^2 \alpha}{k^2}}.$$

CHAPTER VI.

MOTION OF A FLEXIBLE BODY.

THE general term "Flexible Body" includes many other bodies besides strings. The motions treated of in these cases are generally small oscillations, and their discussion will properly form a subject by itself. The reader is therefore referred to any treatise on Sound and to the memoirs of Poisson, Cauchy and others on the subject. In the present chapter only the motion of a perfectly flexible string will be considered.

154. PROP. *To determine the general equations of motion of a string under the action of any forces.*

First. *Let the string be inextensible.*

Let Ox, Oy, Oz, be any axes fixed in space. Let $Xmds$, $Ymds$, $Zmds$, be the impressed forces that act on any element ds of the string whose mass is mds. Let u, v, w, be the resolved parts of the velocities of this element parallel to the axes. Then, by D'Alembert's principle, the element ds of the string is in equilibrium under the action of the forces

$$mds\left(X-\frac{du}{dt}\right),\quad mds\left(Y-\frac{dv}{dt}\right),\quad mds\left(Z-\frac{dw}{dt}\right),$$

and the tensions at its two ends.

Let T be the tension at the point (x, y, z), then $T\dfrac{dx}{ds}$, $T\dfrac{dy}{ds}$, $T\dfrac{dz}{ds}$ are its resolved parts parallel to the axes. The resolved parts of the tensions at the other end of the element will be

$$T\frac{dx}{ds}+\frac{d}{ds}\left(T\frac{dx}{ds}\right)ds,$$

and two similar quantities with y and z written for x.

Hence the equations of motion are

$$\left.\begin{aligned} m\frac{du}{dt}&=\frac{d}{ds}\left(T\frac{dx}{ds}\right)+mX \\ m\frac{dv}{dt}&=\frac{d}{ds}\left(T\frac{dy}{ds}\right)+mY \\ m\frac{dw}{dt}&=\frac{d}{ds}\left(T\frac{dz}{ds}\right)+mZ \end{aligned}\right\} \text{..............(1)}.$$

In these equations the variables s and t are independent. For any the same element of the string, s is always constant, and its path is traced out by variation of t. On the other hand, the curve in which the string hangs at any proposed time is given by variation of s, t being constant. In this investigation s is measured from any arbitrary point, fixed in the string, to the element under consideration.

To find the geometrical equations. We have

$$\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2=1.$$

Differentiating this with respect to t, we get

$$\frac{dx}{ds}\frac{du}{ds}+\frac{dy}{ds}\frac{dv}{ds}+\frac{dz}{ds}\frac{dw}{ds}=0 \text{...............(2)}.$$

The equations (1) and (2) are sufficient to determine x, y, z, and T, in terms of s and t.

155. These equations may be put under another form. Let ϕ, ψ, χ, be the angles made by the tangent at x, y, z,

with the axes of co-ordinates. Then the equations (1) become

$$m\frac{du}{dt}=\frac{d}{ds}(T\cos\phi)+mX\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

with similar equations for v and w.

To find the geometrical equations, differentiate $\cos\phi=\frac{dx}{ds}$ with respect to t;

$$\therefore\ -\sin\phi\frac{d\phi}{dt}=\frac{du}{ds}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Similarly, by differentiating $\cos\psi=\frac{dy}{ds}$ and $\cos\chi=\frac{dz}{ds}$, we get two other similar equations for ψ and χ. Taking these six equations in conjunction with the following

$$\cos^2\phi+\cos^2\psi+\cos^2\chi=1\ldots\ldots\ldots\ldots\ldots(5),$$

we have seven equations to determine u, v, w, ϕ, ψ, χ and T.

If the motion takes place in one plane, these reduce to the four following equations:

$$\left.\begin{aligned}m\frac{du}{dt}&=\frac{d}{ds}(T\cos\phi)+mX\\ m\frac{dv}{dt}&=\frac{d}{ds}(T\sin\phi)+mY\end{aligned}\right\}\ldots\ldots\ldots\ldots(6),$$

$$\left.\begin{aligned}-\sin\phi\frac{d\phi}{dt}&=\frac{du}{ds}\\ \cos\phi\frac{d\phi}{dt}&=\frac{dv}{ds}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

The arbitrary constants and functions which enter into the solutions of these equations must be determined from the peculiar circumstances of each problem.

156. Secondly. *Let the string be elastic.*

The dynamical equations will be the same as before, but the geometrical equations will depend on the elasticity of the string. Let s' be the unstretched length of the arc s.

Then the independent variables are s' and t. We have now

$$\left(\frac{dx}{ds'}\right)^2 + \left(\frac{dy}{ds'}\right)^2 + \left(\frac{dz}{ds'}\right)^2 = \left(\frac{ds}{ds'}\right)^2.$$

Differentiating this with respect to t, we get

$$\frac{dx}{ds'}\frac{du}{ds'} + \frac{dy}{ds'}\frac{dv}{ds'} + \frac{dz}{ds'}\frac{dw}{ds'} = \frac{ds}{ds'} \cdot \frac{d}{dt}\left(\frac{ds}{ds'}\right).$$

But if λ be the elasticity of the string, we have

$$\frac{ds}{ds'} = 1 + \frac{T}{\lambda}.$$

Hence, substituting, we get

$$\frac{dx}{ds'}\frac{du}{ds'} + \frac{dy}{ds'}\frac{dv}{ds'} + \frac{dz}{ds'}\frac{dw}{ds'} = \left(1 + \frac{T}{\lambda}\right) \cdot \frac{1}{\lambda}\frac{dT}{dt}.$$

These two equations together with the dynamical equations (1) will suffice for the determination of u, v, w, s, and T in terms of s' and t.

If we wish to use the equations of motion in the forms (3) or (6), the corresponding geometrical equations (4) or (7) must be modified.

We have

$$\frac{dx}{ds'} = \cos\phi \frac{ds}{ds'} = \cos\phi\left(1 + \frac{T}{\lambda}\right).$$

Hence differentiating, we have

$$\frac{du}{ds'} = -\sin\phi\frac{d\phi}{dt} + \frac{1}{\lambda} \cdot \frac{d}{dt}(T\cos\phi),$$

with similar expressions for $\frac{dv}{ds'}$ and $\frac{dw}{ds'}$.

157. When the motion of the string takes place in one plane, it is often convenient to resolve the velocities along the tangent and normal to the curve.

Let u', v', be the resolved parts of the velocity of the element ds along the tangent and normal to the curve at that element. Let ϕ be the angle the tangent to the element ds makes with the axis of x. Then by Chap. IV. Art. 63, the equations of motion are

$$\left.\begin{aligned} \frac{du'}{dt} - v'\frac{d\phi}{dt} &= X' + \frac{dT}{mds} \\ \frac{dv'}{dt} + u'\frac{d\phi}{dt} &= Y' + \frac{T}{m\rho} \end{aligned}\right\}.$$

The geometrical equations may be obtained as follows. We have

$$u = u'\cos\phi - v'\sin\phi.$$

Differentiating with respect to s, we have by Art. 155,

$$-\frac{d\phi}{dt}\sin\phi = \left(\frac{du'}{ds} - \frac{v'}{\rho}\right)\cos\phi - \left(\frac{dv'}{ds} + \frac{u'}{\rho}\right)\sin\phi,$$

where ρ is the radius of curvature, and is equal to $\dfrac{ds}{d\phi}$. Since the axis of x is arbitrary in position, take it so that the tangent during its motion is parallel to it at the instant under consideration; then $\phi = 0$ and we have

$$0 = \frac{du'}{ds} - \frac{v'}{\rho}.$$

Similarly, by taking the axis of x parallel to the normal,

$$\frac{d\phi}{dt} = \frac{dv'}{ds} + \frac{u'}{\rho}.$$

These four equations are sufficient to determine u', v', ϕ and T in terms of s and t.

If the string be extensible, we have, by differentiating,

$$u = u'\cos\phi - v'\sin\phi,$$

with respect to s', and by Art. 156,

$$-\sin\phi\frac{d\phi}{dt}+\frac{1}{\lambda}\frac{d}{dt}(T\cos\phi)=\left(\frac{du'}{ds'}-\frac{v'}{\rho}\frac{ds}{ds'}\right)\cos\phi$$
$$-\left(\frac{dv'}{ds'}+\frac{u'}{\rho}\frac{ds}{ds'}\right)\sin\phi,$$

whence, by the same reasoning as before, the geometrical equations are

$$\left.\begin{aligned}\frac{1}{\lambda}\frac{dT}{dt}&=\frac{du'}{ds'}-\frac{v'}{\rho}\left(1+\frac{T}{\lambda}\right)\\ \frac{d\phi}{dt}\cdot\left(1+\frac{T}{\lambda}\right)&=\frac{dv'}{ds'}+\frac{u'}{\rho}\left(1+\frac{T}{\lambda}\right)\end{aligned}\right\}.$$

158. Def. When the motion of a string is such that the curve which it forms in space is always equal, similar, and similarly situated to that which it formed in its initial position, that motion may be called steady.

159. Prop. *To investigate the steady motion of an inextensible string.*

It is obvious that every element of the string is animated with two velocities, one due to the motion of the curve in space, and the other to the motion of the string along the curve which it forms in space. Let a and b be the resolved parts along the axes of the velocity of the curve at the time t, and let c be the velocity of the string along its curve.

Then, following the usual notation, we have

$$\left.\begin{aligned}u&=a+c\,.\cos\phi\\ v&=b+c\,.\sin\phi\end{aligned}\right\}\dots\dots\dots\dots\dots\dots\dots(1).$$

Now a, b, c are functions of t only, hence

$$\frac{du}{ds}=-c\sin\phi\,\frac{d\phi}{ds}.$$

But $$\frac{du}{ds} = -\sin\phi\,\frac{d\phi}{dt};$$

$$\therefore\ \frac{d\phi}{dt} = c\,\frac{d\phi}{ds} \quad\text{.......................(2).}$$

Substituting the values of a and b in the equations of motion, Art. 154, we get

$$\left.\begin{aligned}\frac{da}{dt} + \frac{dc}{dt}\cos\phi - c\sin\phi\,\frac{d\phi}{dt} &= X + \frac{d}{ds}\left(\frac{T}{m}\cos\phi\right)\\ \frac{db}{dt} + \frac{dc}{dt}\sin\phi + c\cos\phi\frac{d\phi}{dt} &= Y + \frac{d}{ds}\left(\frac{T}{m}\sin\phi\right)\end{aligned}\right\}.$$

Substituting for $\frac{d\phi}{dt}$, these equations reduce to

$$\left.\begin{aligned}\frac{da}{dt} &= \left(X - \frac{dc}{dt}\cos\phi\right) + \frac{d}{ds}\left\{\left(\frac{T}{m} - c^2\right)\cos\phi\right\}\\ \frac{db}{dt} &= \left(Y - \frac{dc}{dt}\sin\phi\right) + \frac{d}{ds}\left\{\left(\frac{T}{m} - c^2\right)\sin\phi\right\}\end{aligned}\right\} \quad\text{...... (3).}$$

The form of the curve is to be independent of t; hence, on eliminating T, the resulting equation must not contain t. This will not generally be the case unless $\frac{da}{dt}$, $\frac{db}{dt}$, $\frac{dc}{dt}$ are all constants. In any case their values will be determined by the known circumstances of the Problem. The above equations must then be solved, s being supposed to be the only independent variable, and t being constant.

160. If the string move uniformly in space, and the elements of the string glide uniformly along the string, $\frac{da}{dt} = 0$, $\frac{db}{dt} = 0$, $\frac{dc}{dt} = 0$. It will then follow from the above equations, that the form of the string will be the same as if it was at rest, but the tension will exceed the stationary tension by mc^2.

161. Ex. *Let an electric cable be deposited at the bottom of a sea of uniform depth from a ship moving with uniform velocity in a straight line, and let the cable be delivered with a velocity c equal to that of the ship. Find the equation to the curve in which the string hangs.*

The motion may be considered *steady*, and the form of the curve of the string will be always the same.

If the friction of the water on the string be neglected, gravity diminished by the buoyancy of the water will be the only force acting on the string. Hence the form of the travelling curve will be the common catenary, and the tension at any point will exceed the tension in the catenary by the weight of a length of string equal to $\frac{c^2}{g'}$.

Next let the friction of the water on any element of the cable be supposed to vary as the velocity of the element, and to act in a direction opposite to the direction of motion of the element*. Let μ be the coefficient of friction.

Let the axis of x be horizontal, and let x' be the abscissa of any point of the cable measured from the place where the cable touches the ground, in the direction of the ship's motion. Also let s' be the length of the curve measured from the same point. Then $x = x' + ct$, and $s = s' + ct$.

Following the same notation as before, we have

$$X = -\mu u, \qquad Y = -g' - \mu v.$$

But $$u = c - c\cos\phi, \qquad v = -c\sin\phi.$$

Hence the equations (3) become

$$\left.\begin{aligned} 0 &= -\mu c + \mu c\cos\phi + \frac{d}{ds}\left\{\left(\frac{T}{m} - c^2\right)\cos\phi\right\} \\ 0 &= -g' + \mu c\sin\phi + \frac{d}{ds}\left\{\left(\frac{T}{m} - c^2\right)\sin\phi\right\} \end{aligned}\right\}.$$

To integrate these put $\sin\phi = \frac{dy}{ds}$, $\cos\phi = \frac{dx}{ds}$.

* Each element of the string has a motion both along the cable and transversely to it. The coefficients of these frictions are probably not the same, but they have been taken equal in the above investigation.

Hence,

$$\left.\begin{aligned} g'A &= -\mu cs + \mu cx + \left(\frac{T}{m} - c^2\right)\cos\phi \\ g'B &= -\ gs + \mu cy + \left(\frac{T}{m} - c^2\right)\sin\phi \end{aligned}\right\} \qquad\qquad(1),$$

where A and B are two arbitrary constants.

At the point where the cable meets the ground, we must have either $T=0$ or $\phi=0$. For if ϕ be not zero, the tangents at the extremities of an infinitely small portion of the string make a finite angle with each other. Then, if T be not zero, resolving the tensions at the two ends in any direction, we have an infinitely small mass acted on by a finite force. Hence the element will in that case alter its position with an infinite velocity. First, let us suppose that $\phi=0$. Also at the same point, $y=0$ and $s'=0$. Hence $B=-ct$.

Putting $\frac{\mu c}{g'} = e$, we get by division

$$\frac{dy}{dx} = \frac{s'-ey}{A-ex'+es'} \qquad\qquad(2).$$

This is the differential equation to the curve in which the cable hangs.

To solve this equation*, find s' in terms of the other quantities,

$$s' = \frac{A\dfrac{dy}{dx'} - ex'\dfrac{dy}{dx'} + ey}{1 - e\dfrac{dy}{dx'}}.$$

Differentiating, we have

$$\sqrt{1+\left(\frac{dy'}{dx'}\right)^2} = \frac{\dfrac{d^2y}{dx'^2}\cdot(A-ex'+e^2y)}{\left(1-e\dfrac{dy}{dx'}\right)^2}.$$

* *Phil. Mag.* July 1858. The Astronomer Royal on the Mechanical Conditions of the deposit of a Submarine Cable.

Put p for $\frac{dy}{dx}$ where convenient, and put v for $A-ex+e^2y$; the equation then becomes

$$\frac{1}{v}\frac{dv}{dx'}=\frac{-e\frac{dp}{dx'}}{(1-ep)\sqrt{1+p^2}},$$

in which the variables are separated, and the forms are well known. The equation can be integrated a second time, but the result is very long. The arbitrary constant A may have any value, depending on the length of the cable hanging from the ship at the time $t=0$.

The curve in its lower part resembles a circular arc or the lower part of a common catenary. But in its upper part the curve does not tend to become vertical, but tends to approach an asymptote making an angle $\cot^{-1}e$ with the horizon. The asymptote does not pass through the place of touching the ground but below it, its smallest distance being $\frac{A}{e\sqrt{e^2+1}}$, and it also passes below the ship.

If the conditions of the question be such that the tension at the lowest point of the cable is equal to nothing, the tangent to the curve at that point will not necessarily be horizontal. Let λ be the angle this tangent makes with the horizon. Referring to equations (1) we have when

$$x'=0,\ y=0,\ s'=0,\ T=0,\ \text{and } \phi=\lambda.$$

Hence $$A=-\frac{c^2}{g'}\cos\lambda,\ \ B=-\frac{c^2}{g'}\sin\lambda-ct.$$

The differential equation to the curve will now become

$$\frac{dy}{dx'}=\frac{-\frac{c^2}{g'}\sin\lambda+s'-ey}{-\frac{c^2}{g'}\cos\lambda+es'-ex'}\ \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

which can be integrated in the same manner as before. One case deserves notice; viz. when $e = \cot\lambda$. The equation is then evidently satisfied by $y = \frac{1}{e}x'$. The two constants in the integral of (3) are to be determined by the condition that when $x' = 0$, $y = 0$, then $\frac{dy}{dx'} = \tan\lambda$. Both these conditions are satisfied by the relation $y = \frac{1}{e}x'$. Hence this is the required integral. The form of the cable is therefore a straight line, inclined to the horizon at an angle $\lambda = \cot^{-1}e$; and the tension may be found from the formula

$$T = \frac{mg'y}{1+\cos\lambda}.y.$$

162. PROP. *An inelastic string is suspended from two fixed points under the action of gravity so that it hangs in the form of a catenary, the parameter of which is c. Any small disturbance being given to the string in its own plane, it is required to determine the general equation to the small oscillations about the position of rest.*

Let α be the angle the tangent at any point makes with the horizon when the string is at rest, and $\alpha+\phi$ the angle made by the tangent at the same point of the string at the time t.

Let u, v be the velocities of any element ds of the string resolved along the tangent and normal, and T' the tension of the element. Let the mass of a unit of length be taken as the unit. Then the general equations of motion of the string are by Art. 157

$$\left.\begin{aligned} \frac{du}{dt} - v\frac{d\phi}{dt} &= -g\sin(\alpha+\phi) + \frac{dT'}{ds} \\ \frac{dv}{dt} + u\frac{d\phi}{dt} &= -g\cos(\alpha+\phi) + \frac{T'd(\alpha+\phi)}{ds} \end{aligned}\right\}.$$

Let the directrix of the catenary be taken as the axis of x, and let s be measured from that point of the string which coincides with the lowest point of the catenary when the string is in the position of equilibrium. Then the tension when the string is at rest is gy, which is equal to $\dfrac{gc}{\cos\alpha}$.

$$\text{Let } T' = \frac{gc}{\cos\alpha} + T. \quad \text{Also } \tan\alpha = \frac{s}{c};$$

$$\therefore \ \frac{d\alpha}{ds} = \frac{\cos^2\alpha}{c} \text{ and } \frac{dT'}{ds} = \frac{\cos^2\alpha}{c} \cdot \frac{dT'}{d\alpha}.$$

Substituting these values of T' and $\dfrac{dT'}{ds}$, and remembering that in small oscillations we may neglect the squares and products of the small quantities u, v, ϕ, we get

$$\frac{du}{dt} = -g\cos\alpha\,.\,\phi + \frac{\cos^2\alpha}{c}\frac{dT}{d\alpha} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (1),$$

$$\frac{dv}{dt} = g\sin\alpha\,.\,\phi + g\cos\alpha\,.\,\frac{d\phi}{d\alpha} + \frac{\cos^2\alpha}{c}\,.\,T \ldots\ldots \ (2).$$

We have also by Art. 157 the two geometrical equations,

$$\left.\begin{aligned} \frac{du}{ds} - \frac{v}{\rho} &= 0 \\ \frac{dv}{ds} + \frac{u}{\rho} &= \frac{d\phi}{dt} \end{aligned}\right\},$$

where $\dfrac{1}{\rho} = \dfrac{d\alpha}{ds} + \dfrac{d\phi}{ds}$ is the reciprocal of the radius of curvature.

Changing the independent variable to α and neglecting the squares of small quantities, these reduce to

$$\left.\begin{aligned} v &= \frac{du}{d\alpha} \\ \frac{d^2u}{d\alpha^2} + u &= \frac{c}{\cos^2\alpha}\frac{d\phi}{dt} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots \ (3).$$

For the sake of brevity let us put u', v', ϕ'' for $\frac{du}{dt}$, $\frac{dv}{dt}$, $\frac{d^2\phi}{dt^2}$ respectively.

In order to solve these equations, we must eliminate T from the equations (1) and (2). Differentiating the second equation, we get

$$\frac{d^2u'}{d\alpha^2} = g\cos\alpha\,.\,\phi + g\cos\alpha\frac{d^2\phi}{d\alpha^2} + \frac{\cos^2\alpha}{c}\frac{dT}{d\alpha} - \frac{2\cos\alpha\sin\alpha}{c}T.$$

Subtracting equation (1) from this result, we have

$$\frac{d^2u'}{d\alpha^2} - u' = g\cos\alpha\left(\frac{d^2\phi}{d\alpha^2} + 2\phi\right) - \frac{2\cos\alpha\sin\alpha}{c}T.$$

Eliminating T from this by means of (2), we get

$$\cos\alpha\left(\frac{d^2u'}{d\alpha^2} + u'\right) + 2\left(\sin\alpha\frac{du'}{d\alpha} - u'\cos\alpha\right) = g\cos^2\alpha\frac{d^2\phi}{d\alpha^2}$$
$$+ 2g\sin\alpha\cos\alpha\frac{d\phi}{d\alpha} + 2g\phi \;\ldots\ldots\ldots\; (4).$$

But by (3)

$$\frac{d^2u'}{d\alpha^2} + u' = \frac{c}{\cos^2\alpha}\phi'' \;\ldots\ldots\ldots\ldots\ldots\ldots\; (5);$$

$$\therefore\ \sin\alpha\frac{du'}{d\alpha} - u'\cos\alpha = c\int\frac{\sin\alpha}{\cos^2\alpha}\phi''d\alpha,$$

substituting these in (4), we get

$$\frac{c}{\cos\alpha}\phi'' + 2c\int\frac{\sin\alpha}{\cos^2\alpha}\phi''d\alpha = g\left(\cos^2\alpha\frac{d^2\phi}{d\alpha^2} + 2\sin\alpha\cos\alpha\frac{d\phi}{d\alpha} + 2\phi\right).$$

Differentiating again, we have

$$\frac{c}{\cos\alpha}\frac{d\phi''}{dx} + \frac{3c\sin\alpha}{\cos^2\alpha}\phi'' = g\cos^2\alpha\left(\frac{d^3\phi}{dx^3} + 4\frac{d\phi}{d\alpha}\right);$$

$$\therefore\ \frac{\cos^3\alpha\,\dfrac{d\phi''}{d\alpha}+3\cos^2\alpha\sin\alpha\phi''}{\cos^6\alpha}=\frac{g}{c}\left(\frac{d^3\phi}{d\alpha^3}+4\frac{d\phi}{d\alpha}\right);$$

integrating both sides, we have

$$\frac{\phi''}{\cos^3\alpha}=\frac{g}{c}\left\{\frac{d^2\phi}{d\alpha^2}+4\phi+f(t)\right\}.$$

Returning to the original notation, this may be written

$$\frac{d^2\phi}{dt^2}=\frac{g}{c}\cos^3\alpha\left\{\frac{d^2\phi}{d\alpha^2}+4\phi+f(t)\right\}\ \ldots\ldots\ldots\ldots\ (6).$$

This is the general equation to determine the small oscillations of a slack string.

163. The tension of the string will be given by equation (2), but another expression may also be found as follows.

Differentiating (2) and adding the result to (1), we obviously get by (5),

$$\frac{c}{\cos^2\alpha}\frac{d^2\phi}{dt^2}=g\cos\alpha\frac{d^2\phi}{d\alpha^2}+2\frac{\cos^2\alpha}{c}\frac{dT}{d\alpha}+\frac{T}{c}\frac{d\cos^2\alpha}{d\alpha},$$

$$\text{or }\cos\alpha\frac{dT}{d\alpha}-\sin\alpha\,.\,T=\frac{c^2}{2}\left(\frac{1}{\cos^3\alpha}\frac{d^2\phi}{dt^2}-\frac{g}{c}\frac{d^2\phi}{d\alpha^2}\right)$$

$$=\frac{cg}{2}\{4\phi+f(t)\};$$

$$\therefore\ \cos\alpha\,.\,T=\frac{cg}{2}\int\{4\phi+f(t)\}\,d\alpha;$$

$$\therefore\ T=\frac{cg}{2\cos\alpha}\{f(t)\,.\,\alpha+4\int\phi d\alpha\}.$$

164. If the string be so tight that we may neglect the squares of α, we have, since $\tan\alpha=\dfrac{s}{c}$, $s=c\alpha$ and therefore $\dfrac{ds}{d\alpha}=c$. Hence

$$\frac{d\phi}{d\alpha}=c\frac{d\phi}{ds}\ \text{ and }\ \frac{d^2\phi}{d\alpha^2}=c^2\frac{d^2\phi}{ds^2}.$$

The equation (6) now reduces to

$$\frac{d^2\phi}{dt^2} = gc\frac{d^2\phi}{ds^2} + \frac{g}{c}\{4\phi + f(t)\}.$$

To simplify the equation let $\phi = F(t) + \phi'$ and let $F(t)$ be such that

$$\frac{d^2F(t)}{dt^2} = \frac{g}{c}\{4F(t) + f(t)\},$$

then we have

$$\frac{d^2\phi'}{dt^2} = gc\frac{d^2\phi'}{ds^2} + \frac{4g}{c}\phi' \quad\text{.................... (7).}$$

Since c is very great, the first term on the right-hand side is much more important than the second.

165. *An infinite string is suspended in equilibrium in the form of a catenary. A small disturbance being given to it, it is required to determine the motion of the lower part of the string.*

Taking the same notation as before, the equation of motion is

$$\frac{d^2\phi}{dt^2} = \frac{g}{c}\cos^3\alpha\left\{\frac{d^2\phi}{d\alpha^2} + 4\phi + f(t)\right\} \quad\text{............. (6).}$$

The integration of this equation will introduce two arbitrary functions into the value of ϕ. The forms of these functions are determined by the initial values of ϕ and $\frac{d\phi}{dt}$. The form of the function $f(t)$ is determined by the condition that when $\alpha = \frac{\pi}{2}$, $\phi = 0$. The geometrical meaning of $f(t)$ may be found by referring to the last article. It is there shown that if we put $\phi = F(t) + \phi'$, the function $f(t)$ may be eliminated altogether provided α be small. Now $F(t)$ is the same for all the elements of the string, being a function of t only. Hence near the lower part of the string, where α is small, the equation $\phi = F(t)$ is equivalent to a small rotatory motion of the curve as a whole, its form being unchanged.

The values of u and v may be found from equations (3) in page 223,

$$\left.\begin{aligned} v &= \frac{du}{d\alpha} \\ \frac{d^2u}{d\alpha^2} + u &= \frac{c}{\cos^2\alpha}\frac{d\phi}{dt} \end{aligned}\right\} \text{..................(3).}$$

The integration of these equations will introduce two arbitrary functions of t, and we shall have

$$u = \psi(t)\sin\alpha + \chi(t)\cos\alpha + \left\{\begin{array}{l}\text{terms depending on}\\ \text{the value of } \phi,\end{array}\right.$$

$$v = \psi(t)\cos\alpha - \chi(t)\sin\alpha + \left\{\begin{array}{l}\text{terms depending on}\\ \text{the value of } \phi.\end{array}\right.$$

These arbitrary functions are evidently equivalent only to small motions of translation $\chi(t)$ and $\psi(t)$ parallel to x and y respectively, the curve moving as a whole, its form being unchanged. By considering the way in which the string is supported, the values of u and v will be known throughout the motion at some points of the curve. This will enable us to determine the forms of $\chi(t)$ and $\psi(t)$.

We shall assume that the string never departs far from its position of rest. In determining the motion of the lower part of the string we may put $\cos^3\alpha = 1$, and we may therefore use the equation (7) of the last article,

$$\frac{d^2\phi'}{dt^2} = cg\frac{d^2\phi'}{ds^2} + \frac{4g}{c}\phi' \text{ .... (7),}$$

where $\phi = \phi' + F(t)$.

To obtain the integral of this equation, assume

$$\phi' = L\sin(ms + nt);$$

substituting in the equation, we get

$$-n^2 = -gc \, . \, m^2 + \frac{4g}{c};$$

$$\therefore\ n = \pm m\sqrt{gc - \frac{4g}{cm^2}}$$

Let this be written for brevity

$$n = \pm ma.$$

Then
$$\left.\begin{aligned}\phi' &= L \sin m (s + at)\\ \phi' &= M \sin m (s - at)\end{aligned}\right\}$$

are both integrals of the differential equation. Similarly

$$\left.\begin{aligned}\phi' &= L' \cos m (s + at)\\ \phi' &= M' \cos m (s - at)\end{aligned}\right\}$$

are also integrals, where L, M, L', M', m may have any values. The complete value of ϕ' will therefore be a series of terms whose general form is the sum of these four partial integrals, i.e.

$$\left.\begin{aligned}\phi' &= L \sin m (s + at) + L' \cos m (s + at)\\ &\quad + M \sin m (s - at) + M' \cos m (s - at)\,;\\ \therefore\ \frac{1}{a}\frac{d\phi'}{dt} &= Lm \cos m (s + at) - L'm \sin m (s + at)\\ &\quad - Mm \cos m (s - at) + M'm \sin m (s - at)\end{aligned}\right\} \ldots\ldots (8).$$

To determine the constants L, M, L', M' we must have recourse to the initial values of ϕ' and $\dfrac{d\phi'}{dt}$. Let these initial values be expanded in a series of terms of the form

$$\left.\begin{aligned}\phi' &= A \sin ms + A' \cos ms\\ \frac{1}{a}\frac{d\phi'}{dt} &= B \sin ms + B' \cos ms\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (9).$$

Then A, A', B, B' are known quantities. See Todhunter's *Integral Calculus*, Chap. XIII.

These values of ϕ' and $\dfrac{d\phi'}{dt}$ must agree with those given by equations (8) when $t = 0$. Taking only the general term of each, we get

$$\left.\begin{aligned} L+M&=A \\ (L-M)\,m&=B' \end{aligned}\right\}.$$

Hence
$$\left.\begin{aligned} L&=\tfrac{1}{2}A+\frac{B'}{2m} \\ M&=\tfrac{1}{2}A-\frac{B'}{2m} \end{aligned}\right\},$$

with similar expressions for L' and M'. By substituting these in equations (8) we get the general form of ϕ'.

To interpret these expressions, take any term in the expression for ϕ', viz.

$$\phi' = L \sin m\,(s \pm at),$$

where
$$a = \sqrt{gc - \frac{4g}{cm^2}}.$$

This is known to be the expression for the motion of a wave whose length is

$$\lambda = \frac{2\pi}{m},$$

and whose velocity of propagation is

$$a = \pm\sqrt{gc}\,.\sqrt{1-\frac{\lambda^2}{c^2\pi^2}}.$$

Since a is different for different values of λ, it appears that waves of different lengths travel with slightly different velocities.

If the initial value of ϕ' contains any term $A \sin ms$ in which $\lambda = \frac{2\pi}{m}$ is equal to or greater than $c\pi$, then the general expression will contain a similar term. The value of a then becomes imaginary, and the form of that part of the integral must be changed.

Put $a = b\sqrt{-1}$, then

$$\begin{aligned} \phi' &= L \sin m\,(s + bt\sqrt{-1}) \\ &= L\,\{\sin ms \cos(mbt\sqrt{-1}) + \cos ms \sin(mbt\sqrt{-1})\}; \end{aligned}$$

writing for $\sin(mbt\sqrt{-1})$ and $\cos(mbt\sqrt{-1})$ their exponential values, the expression takes the form

$$\phi' = (L\epsilon^{mbt} + M\epsilon^{-mbt}) \sin ms$$
$$+ (L'\epsilon^{mbt} + M'\epsilon^{-mbt}) \cos ms,$$

where L, M, L', M' are different constants from what they were before. Unless $L = 0$, $L' = 0$, the value of ϕ' will soon cease to be small, and the previous investigation will not apply.

If there be a term $A \sin 2\frac{s}{c} + B \cos 2\frac{s}{c}$ in the initial value of ϕ', then $m = 2$ and the quantity a in the corresponding term of the general expression for ϕ' vanishes. The general expression (8) then becomes incomplete, for the terms $L \sin m(s + at)$ and $M \sin m(s - at)$ join into one, and we have no longer the proper number of arbitrary constants. This indicates a change in the form of ϕ. Let us assume

$$\phi = L \sin 2\alpha + M \cos 2\alpha + N,$$

where L, M, N are some unknown functions of t. Substituting in the equations (6), which hold throughout the whole length of the string, we have

$$\frac{d^2L}{dt^2} \sin 2\alpha + \frac{d^2M}{dt^2} \cos 2\alpha + \frac{d^2N}{dt^2} = \frac{g}{c} \cos^3 \alpha \{4N + f(t)\}.$$

But since L, M, N and $f(t)$ do not contain α;

$$\therefore \frac{d^2L}{dt^2} = 0, \quad \frac{d^2M}{dt^2} = 0, \quad \frac{d^2N}{dt^2} = 0, \quad 4N + f(t) = 0.$$

Hence the new term in the expression for ϕ is

$$\phi = (a + bt) \sin 2\alpha + (a' + b't) \cos 2\alpha + a'' + b''t.$$

The values of the arbitrary constants a, b, a', b' are to be found from the initial values of ϕ and $\frac{d\phi}{dt}$.

Also since $\phi = 0$ when $\alpha = \frac{\pi}{2}$ we must have

$$a'' = a', \quad b'' = b'.$$

Unless $b=0$, $b'=0$, the motion soon ceases to be small, and the preceding investigation does not apply.

If the initial value of $\frac{d\phi}{dt}$ is zero, then $b=0$ and $b'=0$, and we have

$$\phi = a \sin 2\alpha + a'(1+\cos 2\alpha).$$

Since this expression does not contain t, it appears that the parts of the string are at rest relatively to each other. This might have been anticipated, for the term $a \sin 2\alpha$ corresponds to a small change in the parameter of the catenary, so that the string is still in the form of a catenary though not the same catenary as before, for if

$$\tan \alpha = \frac{s}{c}, \text{ and } \tan(\alpha+\phi) = \frac{s}{c+\gamma},$$

where s is measured from the lowest point of the catenary and where γ is the small change, then we have by Taylor's theorem

$$\frac{1}{\cos^2 \alpha}\phi = -\frac{s}{c^2}\gamma,$$

or

$$\phi = -\frac{\gamma}{2c}.\sin 2\alpha.$$

The term $a'(1+\cos 2\alpha)$ corresponds to a slip of the string along its length, so that the curve is still in the form of the *same* catenary as before. For if

$$\tan \alpha = \frac{s}{c}, \text{ and } \tan(\alpha+\phi) = \frac{s+\gamma}{c},$$

where γ is the slip, we have

$$\frac{1}{\cos^2 \alpha}\phi = \frac{\gamma}{c};$$

$$\therefore \phi = \frac{\gamma}{2c}.(1+\cos 2\alpha).$$

166. *A finite string of length 2l is fastened at two points in the same horizontal line, whose distance apart is nearly equal to 2l. Find the small oscillations of the string.*

Using the same notation as before, let

$$\phi' = L \sin (ms + nt)$$

be any term in the expression for ϕ', where s is measured from the middle point of the string. The corresponding motion of any element of the string may be found by equations (3) in page 227. Since the string is to be nearly horizontal, α is small and we may put $\cos^2 \alpha = 1$. Hence the equations become

$$\left.\begin{aligned} v &= \frac{du}{d\alpha} \\ \frac{d^2u}{d\alpha^2} + u &= c\,\frac{d\phi}{dt} \end{aligned}\right\}.$$

Integrating these equations we get since $s = c\alpha$ very nearly,

$$\left.\begin{aligned} u &= \psi(t) \sin \frac{s}{c} + \chi(t) \cos \frac{s}{c} + \frac{ncL}{1-m^2} \cos (ms+nt) + c\,F'(t) \\ v &= \psi(t) \cos \frac{s}{c} - \chi(t) \sin \frac{s}{c} - \frac{mnc^2L}{1-m^2} \cos (ms+nt) \end{aligned}\right\}.$$

Since the extremities of the string are fastened to two fixed points, both u and v must vanish when $s = \pm l$.

Hence, putting $s = \pm l$ in the first equation, and subtracting one result from the other, we get

$$2\psi(t) \,.\, \sin \frac{l}{c} + \frac{2L}{1-m^2} \,.\, \sin ml \,.\, \cos nt = 0 \ldots\ldots. (10).$$

Putting $s = \pm l$ in the second equation, and adding the results, we get

$$2\psi(t) \cos \frac{l}{c} + \frac{2mcL}{1-m^2} \,.\, \cos ml \,.\, \cos nt = 0 \ldots\ldots (11).$$

Hence, eliminating $\psi(t)$ between equations (10) and (11),

$$\cos \frac{l}{c} \,.\, \sin ml - mc \sin \frac{l}{c} \cos ml = 0,$$

or $$\tan ml = mc \,.\, \tan \frac{l}{c} = ml \text{ nearly},$$

since c is very great.

This equation, when solved, will give the type of the possible vibrations of the string.

Hence ϕ will consist of a series of terms of the form

$$\phi = L\sin(ms + nt) \text{ ....(1)},$$

where
$$n = \pm m\sqrt{gc - \frac{4g}{cm^2}} \text{ .... (2)},$$

and
$$\tan ml = ml \text{....................... (3)}.$$

If the initial values of ϕ are such as present a term in which the coefficient of s does not satisfy the equation (3), the motion will not be steady. Suppose for example a small disturbance was given to any small part of the string, it will travel along the string, generally in both directions; on reaching either fixed end it will originate a new reflected disturbance, which will travel back and be again reflected at the other end, and so on. Thus there will be discontinuity in the expression for ϕ. See Poisson's *Mécanique*, Art. 485.

167. PROP. *A string is in equilibrium in any curve in one plane, under the action of any forces. Supposing the string to be cut at any proposed point, to determine the tension at any other point when the string is just beginning to move.*

Let Pds, Qds be the resolved parts of the forces along the tangent and normal to any element ds.

Let u, v be the velocities of the element along the tangent and normal. Then the equations of motion are by Art. 157

$$\frac{du}{dt} - v\frac{d\phi}{dt} = P + \frac{dT}{ds} \text{ ....(1)},$$

$$\frac{dv}{dt} + u\frac{d\phi}{dt} = Q + \frac{T}{\rho} \text{ ....(2)},$$

where T is the tension, ρ the radius of curvature, and ϕ the angle the tangent makes with any fixed straight line. The geometrical equations are

$$\frac{du}{ds} - \frac{v}{\rho} = 0 \text{ ....(3)},$$

$$\frac{dv}{ds} + \frac{u}{\rho} = \frac{d\phi}{dt} \text{ ....(4)}.$$

Differentiating (1) and multiplying (2) by $\frac{1}{\rho}$, we get

$$\left.\begin{aligned}\frac{d^2u}{ds\,dt}-v\frac{d^2\phi}{ds\,dt}-\frac{dv}{ds}\frac{d\phi}{dt}=\frac{dP}{ds}+\frac{d^2T}{ds^2}\\ \frac{1}{\rho}\frac{dv}{dt}+\frac{u}{\rho}\frac{d\phi}{dt}=\frac{Q}{\rho}+\frac{T}{\rho^2}\end{aligned}\right\}\text{.........} (5).$$

But by differentiating (3) we have, since $\frac{1}{\rho}=\frac{d\phi}{ds}$,

$$\frac{d^2u}{ds\,dt}-v\frac{d^2\phi}{ds\,dt}-\frac{1}{\rho}\frac{dv}{dt}=0\text{.................}(6).$$

Hence, subtracting the second of equations (5) from the first, we have by (4) and (6)

$$\frac{d^2T}{ds^2}-\frac{T}{\rho^2}+\frac{dP}{ds}-\frac{Q}{\rho}=-\left(\frac{d\phi}{dt}\right)^2.$$

In the beginning of the motion just after the string has been cut we may reject the squares of small quantities, hence $\left(\frac{d\phi}{dt}\right)^2$ may be rejected. Hence we have

$$\frac{d^2T}{ds^2}-\frac{T}{\rho^2}=-\frac{dP}{ds}+\frac{Q}{\rho}.$$

This is the general equation to determine the tension of a string just after it has been cut.

The two arbitrary constants introduced in the solution of this equation are to be determined by the circumstances of the case. If both ends of the string are free, we must have $T=0$ at both ends. If, again, one end be attached to a small ring without inertia, which can slide freely along a given curve, then the velocity of the element attached to the ring resolved in the direction of the normal to the curve must be zero.

168. Ex. *A string is in equilibrium in the form of a circle about a centre of force in the centre. If the string be now cut at any point A, prove that the tension at any point P is instantaneously changed in the ratio of* $1-\frac{\epsilon^{\pi-\theta}+\epsilon^{-(\pi-\theta)}}{\epsilon^{\pi}+\epsilon^{-\pi}}:1$, *where θ is the angle subtended at the centre by the arc AP.*

Let F be the central force, then $P=0$, and $Q=-F$. Let a be the radius of the circle. Then the equation becomes

$$\frac{d^2T}{ds^2}-\frac{T}{a^2}=-\frac{F}{a}.$$

Let s be measured from the point A towards P, then $s=a\theta$; also F is independent of s. Hence we have

$$T=Fa+A\epsilon^{\theta}+B\epsilon^{-\theta}.$$

To determine the arbitrary constants A and B we have the condition $T=0$ when $\theta=0$ and $\theta=2\pi$;

$$\therefore\ T=Fa.\left\{1-\frac{\epsilon^{(\pi-\theta)}+\epsilon^{-(\pi-\theta)}}{\epsilon^{\pi}+\epsilon^{-\pi}}\right\}.$$

But just before the string was cut

$$T=Fa.$$

Hence the result given in the question follows.

EXAMPLES.

1. A heavy elastic ring, whose length when unstretched is given, is stretched round a circular cylinder. The cylinder is suddenly annihilated, find the time which the ring will take to collapse to its natural length.

2. A homogeneous light inextensible string is attached at its extremities to two fixed points, and turns about the straight line joining those points with uniform angular velocity. Find the form of the string supposing its figure permanent.

Result. Let the straight line joining the fixed points be the axis of x, then the form of the string is a plane curve whose equation is $c^2+\left(\frac{dy}{dx}\right)^2=(a-y)^2$ where a and c are two constants.

3. A uniform endless chain is in equilibrium under the action of forces which depend only on the position of the particle acted on. Every element has an equal velocity communicated to it in the direction of the tangent to the chain at that element, prove that the form of the chain will not be altered by the motion.

4. Let a cable be delivered with velocity c' from a ship moving with uniform velocity c in a straight line on the surface of a sea of uniform depth. If the resistance of the water to the cable be proportional to the square of the velocity, the coefficient, B, of resistance for longitudinal motion being different from the coefficient A, for lateral motion, prove that the cable may take the form of a straight line making an angle λ with the horizon, such that $\cot^2\lambda = \sqrt{e^4 + \frac{1}{4}} - \frac{1}{2}$, where e is the ratio of the speed of the ship to the terminal velocity of a length of cable falling laterally in water. Prove also that the tension will be found from the equation

$$T = y - \frac{B}{A} e^2 \left(\frac{c'}{c} - \cos\lambda\right)^2 \frac{y}{\sin\lambda}.$$

5. If $x + \xi$, $y + \eta$ be the co-ordinates of that element of the string in Art. 165, which when at rest was situated at the point (x, y), and if the undisturbed form of the string be a catenary and the disturbance be small, prove that

$$\frac{d\xi}{ds} = -\frac{2s}{c}\phi, \quad \frac{d\eta}{ds} = \phi,$$

where ϕ is the angle between the tangents at (x, y) and $(x + \xi, y + \eta)$.

6. A heavy string is suspended from one extremity, and being slightly disturbed makes small oscillations about the vertical. Find the form of the curve in which it must be placed at rest at the time $t = 0$, in order that every point of the string may reach the vertical at the same time.

Result. If the fixed point be the origin and the axis of x be drawn vertically downwards, the equation to the string at the time t is $y = \phi(x)\cos\sqrt{cg}\,t$, where ϕ is to be determined from the equation $(l-x)\frac{d^2\phi}{dx^2}-\frac{d\phi}{dx}+c\phi=0$, and c is any arbitrary constant depending on the initial form of the curve.

7. A heavy string is attached to two points in the same horizontal line, so that when in equilibrium the string hangs in the form of a catenary which does not differ very much from a straight line. Find the form of the curve in which it must be placed at rest at the time $t=0$, in order that every point of the string may reach the catenary at the same time. If $2l$ be the length of the string, and T the time of a small oscillation, prove that

$$T=2\pi\sqrt{\frac{cl}{g(c\pi-4l)}}.$$

8. A chain, having initially the form of a closed plane curve very nearly a circle, whirls in its own plane round its centre of gravity; determine the motion so far as to form the linear partial differential equation on which it depends.

9. The extreme links of a uniform chain can slide freely on two given curves in a vertical plane, and the whole is in equilibrium under the action of gravity. Supposing the chain to break at any point, prove that the initial tension at any point is $T=y(A\phi+B)$, where y is the altitude of the point above the directrix of the catenary, ϕ the angle the tangent makes with the horizon, and A, B two arbitrary constants. Explain how the constants are to be determined.

10. A string is wound round the under part of a vertical circle and is supported in equilibrium at the ends of a horizontal diameter by two forces. The circle being suddenly removed, prove that the tension at the lowest point is instantly decreased in the ratio $4:\epsilon^{\pi}+\epsilon^{-\pi}$.

CHAPTER VII.

MOTION OF A SYSTEM OF RIGID BODIES.

Sect. I. *Conservation of Areas.*

169. The general equations of motion of a system of rigid bodies have been deduced in Chap. II. from D'Alembert's Principle. The equations thus obtained are of the second order, and the integration of these equations constitutes the chief difficulty in determining the motion of the system. Certain general methods have been proposed, and we may, if we please, use these in solving the equations of motion as shown in Chap. IV. But these methods always lead to equations of the same form; hence, having once noticed this form, we may make certain rules to write down these integrals at once, and thus avoid the equations of the second order either altogether or in part. These rules are called Principles, and are three in number, viz. the Conservation of Areas, the Conservation of the Momentum of the Centre of Gravity, and the Vis Viva. The first two will be considered under one head.

170. Prop. *If a system of particles be in motion under the action of forces which have no moment about a certain fixed straight line, then the sum of the products of the mass of each particle and the area which its projection on any plane perpendicular to that line describes about the line is proportional to the time during which the motion is considered.*

Let this straight line be taken as the axis of z, and let the plane of xy be the plane on which the areas are described. Let m be the mass of any particle of which the co-ordinates are x, y, z, and let X, Y, Z be the accelerating forces on the particle. Then the equation

$$\Sigma m \left(x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2} \right) = \Sigma m (xY - yX)$$

becomes

$$\Sigma m \left(x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2} \right) = 0.$$

Integrating, we have

$$\Sigma m \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) = h,$$

where h is some constant. Let A be the area described in the time t by the radius vector of the projection of m. Then by Differential Calculus we know that

$$2 \frac{dA}{dt} = x \frac{dy}{dt} - y \frac{dx}{dt}.$$

Hence the equation becomes

$$\Sigma \left(m \frac{dA}{dt} \right) = \frac{1}{2} h;$$

$$\therefore \Sigma (mA) = \frac{1}{2} ht + c,$$

but when $t = 0$ each term of the sum on the left hand vanishes; therefore $c = 0$.

$$\therefore \Sigma (mA) = \frac{1}{2} ht.$$

Hence the proposition is proved.

The intersection of the line about which the moment of the forces is zero with the plane on which the areas are traced is called the *pole of areas*. The straight line itself is called the *axis of areas*. The quantity h is called the area conserved in two units of time, or sometimes more simply *the area conserved*.

171. Prop. *If a system of particles be in motion under the action of forces which have no component along a certain fixed straight line, then the sum of the products of the mass of each particle and its velocity resolved along this straight line is constant.*

Let this straight line be taken as the axis of z. Let m be the mass of any particle of which the co-ordinates are x, y, z; and let X, Y, Z be the accelerating forces on the particle. Then the equation

$$\Sigma\left(m\frac{d^2z}{dt^2}\right) = \Sigma\,(mZ)$$

becomes

$$\Sigma\left(m\frac{d^2z}{dt^2}\right) = 0.$$

Integrating, we have

$$\Sigma\left(m\frac{dz}{dt}\right) = c,$$

where c is some constant.

If v be the velocity of the particle whose mass is m, and if α, β, γ be the angles its direction of motion makes with the axes respectively, the equation may be written

$$\Sigma\,(mv\cos\gamma) = c.$$

This principle has been deduced from the equations of the motion of translation of the system, and the conservation of areas from the equations of rotation. The first is called the principle of the *conservation of linear momentum*, and the second should be called the principle of the *conservation of angular momentum.*

172. Cor. From this proposition we may also infer that the velocity of the centre of gravity of the whole system resolved parallel to the above straight line is constant.

For if $\bar{x}, \bar{y}, \bar{z}$ be the co-ordinates of the centre of gravity, we have

$$M\frac{d\bar{z}}{dt} = \Sigma\left(m\frac{dz}{dt}\right)$$
$$= c.$$

Hence $\frac{d\bar{z}}{dt}$, the velocity of the centre of gravity, is constant.

173. It is obvious that the moment of all the internal forces of any system of particles about any straight line is always zero. Thus the mutual attractions of two particles being equal and opposite, their moments will be also equal and opposite. So also any impacts between the particles of the system, or any explosions, will not affect the truth of the principle. Hence if no *external* forces act on the system, the areas conserved on any plane and about any pole in that plane will be proportional to the time. But these areas will be different for different planes, and for different poles in each plane.

174. PROP. III. *In a system acted on by no external forces, to compare the areas conserved on different planes passing through the same pole.*

Let this point be taken as the origin, and let h_1, h_2, h_3 be the areas conserved on the co-ordinate planes yz, zx, xy. Then the area conserved on any other plane will be the sum of the projections of the areas conserved on the co-ordinate planes. Let H be the area conserved on the plane whose direction-cosines are L, M, N, then

$$H = Lh_1 + Mh_2 + Nh_3.$$

Let $h^2 = h_1^2 + h_2^2 + h_3^2$, then by this formula it is evident that the area conserved on the plane whose direction-cosines are

$$l = \frac{h_1}{h}, \quad m = \frac{h_2}{h}, \quad n = \frac{h_3}{h},$$

is equal to h; and

$$H = h\,(Ll + Mm + Nn).$$

Let θ be the angle between these two planes, then

$$\cos\theta = Ll + Mm + Nn.$$

$$\therefore\ H = h\,.\cos\theta.$$

From this it is evident that H is a maximum when $\theta = 0$. Hence

With any given point for pole there is a certain plane on which the area conserved is a maximum, and the direction-cosines of this plane being constants, this plane is fixed throughout the whole motion. This plane is called the Invariable Plane, and its normal is called the Invariable Line.

Also, the area conserved on any plane is equal to the area conserved on the invariable plane multiplied by the cosine of the angle between the planes.

175. Prop. IV. *To compare the areas conserved on the same plane about different poles.*

Let the plane on which the areas are described be taken as the plane of xy; let $\bar{x}, \bar{y}$ be the co-ordinates of the projection of the centre of gravity. Let $x = \bar{x} + x', y = \bar{y} + y'$ be the co-ordinates of any particle of mass m. Then

$$\Sigma m \left(x' \frac{dy'}{dt} - y' \frac{dx'}{dt} \right)$$

is the area conserved about the centre of gravity considered as a fixed point. Also

$$\Sigma m \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right)$$

is the area conserved about the origin. The difference between these two is

$$\left(\bar{x} \frac{d\bar{y}}{dt} - \bar{y} \frac{d\bar{x}}{dt} \right) \Sigma m + \bar{x} \Sigma m \frac{dy'}{dt} - \bar{y} \Sigma m \frac{dx'}{dt}$$
$$+ \frac{d\bar{y}}{dt} \Sigma m x' - \frac{d\bar{x}}{dt} \Sigma m y'.$$

But $\Sigma m x' = 0$, $\Sigma m y' = 0$, and therefore

$$\Sigma m \frac{dx'}{dt} = 0, \quad \Sigma m \frac{dy'}{dt} = 0.$$

Hence the above difference reduces to

$$\left(\bar{x}\frac{d\bar{y}}{dt} - \bar{y}\frac{d\bar{x}}{dt}\right)\Sigma m.$$

But this is the area conserved by the whole mass collected at the centre of gravity about the origin. Hence we have this theorem:

The area conserved about any origin is equal to the area conserved by the whole system about the projection of the centre of gravity plus the area conserved by the whole mass collected at the centre of gravity round the original pole; or, in other words, it is equal to the area conserved by the motion of rotation plus the area conserved by the translation.

This proposition might also have been deduced from Art. 5.

176. PROP. V. *To compare the positions of the invariable plane at different points of a system acted on by no external forces.*

If the centre of gravity be initially at rest, it will remain fixed throughout the motion (Chap. II. Art. 33), and therefore the area conserved by it about any point is zero. Hence the areas conserved about all poles in any plane are the same, and equal to that conserved on a parallel plane through the centre of gravity. It follows also that the invariable plane corresponding to any pole is parallel to the invariable plane at the centre of gravity.

If the centre of gravity be in motion, it moves in a straight line with uniform velocity. Let this straight line be taken as the axis of x, and let the plane of xz be taken so as to contain the invariable line at the centre of gravity. Let V be the uniform velocity of the centre of gravity; M the mass of the body; h_1, h_3 the areas conserved about the axes of x and z.

Let x, y, z be the co-ordinates of any point P, then the areas conserved about parallels to the axes at P are by Art. 175 respectively

$$H_1 = h_1, \quad H_2 = -MVz,$$

$$H_3 = h_3 + MVy.$$

Hence the direction-cosines of the invariable plane at P are

$$l = \frac{h_1}{H}, \quad m = \frac{-MVz}{H}, \quad n = \frac{h_3 + MVy}{H};$$

where $$H^2 = h_1^2 + MV^2z^2 + (h_3 + MVy)^2.$$

Now l, m, n and H are constants if y and z are constant; hence the position of the invariable plane and the area conserved upon it are constant for all poles situated in any straight line parallel to the direction of motion of the centre of gravity.

Again, H is a minimum when $z = 0$ and $h_3 + MVy = 0$. In this case $l = 1$, $m = 0$, $n = 0$, or the invariable plane is perpendicular to the direction of motion of the centre of gravity.

177. The area conserved on the invariable plane at any other point Q, whose co-ordinates are x', y', z', is given by

$$H'^2 = h_1^2 + M^2V^2z'^2 + (h_3 + MVy')^2.$$

$$\text{But } H^2 = h_1^2, \text{ and } h_3 + MVy = 0;$$

$$\therefore\ H'^2 = H^2 + M^2V^2\{z'^2 + (y' - y)^2\}.$$

Through the point at which the area H conserved on the invariable plane is a minimum, draw a straight line parallel to the direction of motion of the centre of gravity. Let r be the distance of Q from this straight line. Then the area H' conserved on the invariable plane at Q is given by

$$H'^2 = H^2 + V^2 . r^2.$$

178. Prop. VI. *In a system acted on by no external forces to determine in what cases the area conserved on a given plane about a moving pole in a given time is constant.*

Let the given plane be taken as the plane of xy. Let p, q be the co-ordinates of the moving pole referred to any fixed axes of co-ordinates Ox, Oy.

Let x, y be the co-ordinates of any particle m, and let

$$x = p + x', \quad y = q + y'.$$

Then the area conserved about the moving pole is

$$\Sigma m \left(x' \frac{dy'}{dt} - y' \frac{dx'}{dt}\right) = h',$$

and that about the fixed pole,

$$\Sigma m \left(x \frac{dy}{dt} - y \frac{dx}{dt}\right) = h;$$

$$\therefore\ h' = \Sigma m \left\{(x-p) \frac{d(y-q)}{dt} - (y-q) \frac{d(x-p)}{dt}\right\}$$

$$= h + M(p-\bar{x}) \frac{dq}{dt} - M(q-\bar{y}) \frac{dp}{dt}$$

$$- Mp \frac{d\bar{y}}{dt} - Mq \frac{d\bar{x}}{dt}.$$

Hence the area conserved about the moving pole is the same as the area conserved about the fixed pole, when

$$(p-\bar{x}) \frac{dq}{dt} - (q-\bar{y}) \frac{dp}{dt} = p \frac{d\bar{y}}{dt} - q \frac{d\bar{x}}{dt}.$$

There are two cases in which this condition can be conveniently satisfied.

First. When the moving pole is the centre of gravity; for then $p = \bar{x}$, $q = \bar{y}$, and if the origin be taken in the line of motion of the centre of gravity $\frac{d\bar{y}}{d\bar{x}} = \frac{\bar{y}}{\bar{x}}$.

Secondly. When the moving pole moves uniformly along a straight line parallel to the line of motion of the centre of gravity and with the same velocity as the centre of gravity. For then

$$\frac{dp}{dt} = \frac{d\bar{x}}{dt}, \quad \frac{dq}{dt} = \frac{d\bar{y}}{dt}, \text{ and } \frac{dq}{dp} = \frac{\bar{y}}{\bar{x}}.$$

179. Prop. VII. *To find the area conserved on any plane about any pole by a rigid body in motion.*

First. Let the body be entirely in the plane on which the areas are conserved. Let r, θ be the polar co-ordinates of the centre of gravity referred to the pole of the areas as origin. Let ω be the angular velocity of the body about its centre of gravity, Mk^2 the moment of inertia about the same point.

Then the area conserved is equal to the area conserved about the centre of gravity plus that conserved by the whole mass collected at the centre of gravity. The first of these is clearly $\Sigma m \left(r'^2 \frac{d\theta'}{dt}\right)$, where r', θ' are the co-ordinates of any particle m of the body referred to the centre of gravity as origin. But since the body is a rigid system, $\frac{d\theta'}{dt}$ is the same for every particle and equal to ω. Hence we have

$$\begin{aligned} \Sigma m \left(r'^2 \frac{d\theta'}{dt}\right) &= \Sigma m r'^2 \,.\, \omega \\ &= Mk^2 \,.\, \omega \quad \ldots\ldots\ldots\ldots\ldots\ldots (1). \end{aligned}$$

The area conserved by the whole mass M collected at the centre of gravity is $M \,.\, r^2 \frac{d\theta}{dt}$. Hence the whole area conserved about the origin is

$$= Mr^2 \frac{d\theta}{dt} + Mk^2 \omega \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

There is another useful form into which this expression may be put. If v be the linear velocity of the centre of gravity, and p the perpendicular from the origin on its direction of motion, then

$$r^2 \frac{d\theta}{dt} = vp.$$

Hence the whole area conserved about the origin is

$$= Mvp + Mk^2\omega \quad \text{.....................} \quad (3).$$

Secondly. Let the body be in motion in space of three dimensions. Let the plane on which the areas are conserved be taken as the plane of xy, and the pole as the origin. Let x, y, z be the co-ordinates of any particle m; $\bar{x}, \bar{y}, \bar{z}$ be the co-ordinates of the centre of gravity G.

Let $\omega_x, \omega_y, \omega_z$ be the angular velocities of the body about the axes of x, y, z; A', B', C' the moments of inertia about these axes.

The area conserved is

$$= \Sigma m \left(x \frac{dy}{dt} - y \frac{dx}{dt}\right).$$

But

$$\left.\begin{aligned} \frac{dx}{dt} &= \omega_y z - \omega_z y \\ \frac{dy}{dt} &= \omega_z x - \omega_x z \end{aligned}\right\}.$$

Hence, substituting, the area conserved is

$$\Sigma m \{(x^2 + y^2)\, \omega_z - \omega_y\, yz - \omega_x\, xz\}$$
$$= C' \,.\, \omega_z - (\Sigma myz) \,.\, \omega_y - (\Sigma mxz)\, \omega_x \quad \text{............} \quad (4).$$

If the axis of z be a principal axis, or if it be the axis of instantaneous rotation, this takes the simple form

$$\text{area conserved} = C'\omega_3 \quad \text{...............} \quad (5).$$

If axes Gx', Gy', Gz' be taken through the centre of gravity parallel to the axes of co-ordinates, the above expression may be put into the form

$$M\left(\bar{x}\frac{d\bar{y}}{dt}-\bar{y}\frac{d\bar{x}}{dt}\right)+C\omega_z-(\Sigma m'y'z')\,\omega_y-(\Sigma mx'z')\,\omega_x\ldots\ldots\ldots(6),$$

where A, B, C are the moments of inertia about the axes Gx', Gy', Gz'.

Let the point O be fixed in the body, and let ω_1, ω_2, ω_3 be the angular velocities of the body about the principal axes at O, and let A_1, B_1, C_1 be the principal moments of inertia.

The areas conserved on the principal planes are respectively $A_1\omega_1$, $B_1\omega_2$, $C_1\omega_3$.

The area conserved on any other plane is the sum of the projections of these three areas. Let l, m, n be the direction-cosines of the normal to the plane of xy referred to the principal axes at O. Then the area conserved on this plane will be

$$A_1\omega_1 l+B_1\omega_2 m+C_1\omega_3 n\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

180. Ex. 1. *To find the invariable plane at the centre of gravity of the solar system.*

Let the centre of gravity be taken as the origin, and let the system be referred to any rectangular axes Gx, Gy, Gz. Let ω be the angular velocity of any planet about its axis, and Mk^2 its moment of inertia about the axis. Let α, β, γ be the direction-cosines of the axis. The axis of revolution and two perpendicular axes form a system of principal axes at the centre of gravity. The area conserved by the planet on a plane perpendicular to the axis is $Mk^2\omega$, and on any plane through the axis, zero. Hence the whole area conserved on the plane of xy round the centre of gravity is by Art. 179

$$Mk^2\omega\cos\alpha.$$

Let r be the distance of the centre of the planet from the centre of gravity of the solar system, $\frac{d\theta}{dt}$ the angular velocity of r in the plane of the instantaneous orbit of the planet. Then the area conserved by the centre of the planet on the plane of xy is

$$Mr^2 \frac{d\theta}{dt} \cos \alpha',$$

where α' is the inclination of the orbit to the plane of xy.

Hence the whole area conserved by the planet on the plane of xy is

$$h_1 = Mr^2 \frac{d\theta}{dt} \cos \alpha' + Mk^2\omega \cos \alpha.$$

The values of h_2, h_3 may be found in a similar manner, and thence the position of the invariable plane.

181. Ex. 2. *If three particles of masses m, m', m'', attracting each other, start from rest, prove that at any instant the tangents to their paths will meet in a point, and that if parallels to their directions of motion be drawn so as to form a triangle, the momenta of the several particles are as the sides of that triangle.*

Let v, v', v'' be the velocities of the particles. The area conserved by any particle of mass m moving with velocity v is mvp, where p is the length of the perpendicular from the origin on the direction of motion. Hence by Art. 170

$$mv\,.\,p + m'v'\,.\,p' + m''v''\,.\,p'' = h,$$

where h is some constant. But in the beginning of the motion

$$v = 0,\ v' = 0,\ v'' = 0\,;\ \therefore\ h = 0.$$

Therefore if three forces represented by mv, $m'v'$, $m''v''$ were to act along the directions of motion, the sum of their moments about every point would be zero. Therefore these forces are in equilibrium, and if a triangle be constructed by drawing lines parallel to their directions, the forces will be proportional to the sides of that triangle.

Also the three forces must meet in a point, hence the three particles are always moving to or from some point O. But this point is not in general a fixed point throughout the motion.

If there be n particles, it may be shown in the same way that the n forces represented by mv, $m'v'$, &c. are in equilibrium; and therefore if parallels be drawn to the directions of motion so as to form a polygon, the momenta of the particles are proportional to the sides of that polygon. It will not however be necessarily true that all the n directions of motion meet in a point.

If F, F', F'' be the resultant attraction on the three particles, the lines of action of F, F', F'' also meet in a point. For let X, Y, Z be the actions between the particles $m'm''$, $m''m$, mm', taken in order. Then F is the resultant of $-Y$ and Z; F' of X and $-Z$; F'' of Y and $-X$. Hence the three forces F, F', F'' are in equilibrium*, and therefore their lines of action must meet in a point O'. Also the magnitude of each is proportional to the sine of the angle between the directions of the other two. This point is not generally fixed, and does not coincide with O.

If the law of attraction be proportional to the distance, the two points O, O' coincide with the centre of gravity G, and are fixed in space throughout the motion. For it is a known proposition in Statics that with this law of attraction, the whole attraction of a system of particles on one of the particles is the same as if the whole system were collected at its centre of gravity. Hence O' coincides with G. Also, since each particle starts from rest, the initial velocity of the centre of gravity is zero, and therefore, by Art. 33, G is a fixed point. Again, since each particle starts from rest and is urged towards a fixed point G, it will move in the straight line joining its initial position with G. Hence O coincides with G. When the law of attraction is proportional to the distance, it is proved in Dynamics of a Particle, that the time of reaching the centre of force from a position of rest is independent of the distance of that position of rest. Hence all the particles of the system will reach G at the same time, and meet there. If Σm be the sum of the masses, measured by their attractions in the usual manner, this time is known to be

$$\frac{1}{4}\frac{2\pi}{\sqrt{\Sigma m}}.$$

182. The principle of the conservation of areas may be

* This proof is merely an amplification of the following. The three forces F, F', F'', being the internal re-actions of a system of three bodies, are in equilibrium by D'Alembert's Principle.

applied to any system under the influence of forces which have no moment about one straight line. We may therefore make use of the principle in the following cases.

First. If no external forces act on the system the principle will apply with any straight line as an axis of areas. Thus, supposing the solar system to be under the influence of no external force, the whole area conserved on any plane about any pole is constant. Let r be the distance of any planet, treated as a particle, from any fixed straight line arbitrarily chosen, and let ω be its angular velocity round this straight line. Then, by the principle,

$$\Sigma mr^2\omega = h \quad \text{..........................(1)}.$$

To illustrate the principle further, let us suppose the bodies composing the system to become rigidly connected. The mutual impulsive actions between the particles will be equal and opposite, and therefore will not appear in the equation furnished by the principle. Hence the equation (1) will still hold. The motion of the centre of gravity by Art. 33 will also not be affected by these actions, and if initially at rest it will continue at rest, if initially in motion it will continue to move in the same straight line as before, and with the same velocity. Let $\Sigma mr^2\omega$ be the area conserved by the system before it became rigid about any axis Gz, through the centre of gravity. Let Mk^2 be the moment of inertia of the system about this axis at the moment after it became rigid, and let Ω be the angular velocity of the rigid system. Then we have, by Art. 178,

$$\left.\begin{aligned} \Sigma mr^2\omega &= h \\ Mk^2\Omega &= h \end{aligned}\right\}.$$

Hence
$$\Omega = \frac{\Sigma mr^2\omega}{Mk^2}.$$

This gives the initial angular velocity of the rigid system about any straight line through the centre of gravity. Hence the angular velocities about three co-ordinate axes may be found, and thence the whole motion.

Secondly. If the resultant of all the forces which act on the system be always normal to a fixed plane, the principle will apply with any straight line perpendicular to this plane as axis of areas. Thus if a system of bodies be under the action of gravity, the whole area conserved on any horizontal plane is constant.

Thirdly. If the resultant of all the forces which act on the system always passes through a fixed point, the principle will apply with any straight line through that point as axis of areas. Thus if a particle m be moving under the action of a centre of force, we have

$$mr^2 \frac{d\theta}{dt} = h;$$

where r is its distance from the centre of force, and $\frac{d\theta}{dt}$ its angular velocity. This is the ordinary equation of motion round a centre of force obtained in Dynamics of a Particle. We may now extend this equation to the case of a system of particles moving round a centre of force and attracting each other according to any law of force. The equation is then

$$\Sigma mr^2 \frac{d\theta}{dt} = h.$$

Fourthly. If the resultant of all the forces always passes through a straight line, the principle will apply if areas be conserved about this straight line as axis of areas. Thus if a system of particles move under the action of *two* centres of force, the equation of areas will hold if conserved about the straight line joining the two centres of force.

183. PROP. VIII. *A rigid body is moving freely in a known manner under the action of no external forces. Suddenly either a straight line or a point in the body becomes fixed. To determine the subsequent motion.*

It is obvious that all the external actions on the body pass through the fixed straight line or the fixed point. Hence the principle of the Conservation of Areas will apply if the plane of areas be either perpendicular to the straight line or pass through the given point.

First. Let a straight line suddenly become fixed. Let it be taken as the axis of z.

Let MK^2 be the moment of inertia of the body about the axis of z, and Ω the angular velocity after the straight line has become fixed. Suppose that the body when moving freely was turning with angular velocities ω_x, ω_y, ω_z about three straight lines Gx', Gy', Gz' through the centre of gravity parallel to the axes of co-ordinates. And let the co-ordinates of the centre of gravity be $\bar{x}$, $\bar{y}$, $\bar{z}$.

Then, by the principle of the Conservation of Areas (Art. 179),

$$C'\omega_z - (\Sigma mz'x')\,\omega_x - (\Sigma mz'y')\,\omega_y + M\left(\bar{x}\frac{d\bar{y}}{dt} - \bar{y}\frac{d\bar{x}}{dt}\right) = MK^2 . \Omega,$$

where C' is the moment of inertia of the body about Gz', and $\Sigma mz'x'$, $\Sigma mz'y'$ are calculated with reference to the axes Gx', Gy', Gz'.

Secondly. Let a point O in the moving body be suddenly fixed in space. Take any three rectangular axes Ox, Oy, Oz, and three parallel axes Gx', Gy', Gz' through the centre of gravity G. Let ω_x, ω_y, ω_z be the known angular velocities of the body about the axes Gx, Gy, Gz before the point O became fixed, Ω_x, Ω_y, Ω_z the unknown angular velocities about Ox, Oy, Oz after O became fixed.

Then, following the same notation as before, we have

$$A\omega_x - (\Sigma m\,x'y')\,\omega_y - (\Sigma m\,x'z')\,\omega_z + \Sigma m\left(\bar{y}\frac{d\bar{z}}{dt} - \bar{z}\frac{d\bar{y}}{dt}\right)$$
$$= A'\Omega_x - (\Sigma m\,xy)\,\Omega_y - (\Sigma m\,xz)\,\Omega_z.$$

$$B\omega_y - (\Sigma m\,y'z')\,\omega_z - (\Sigma m\,y'x')\,\omega_x + \Sigma m\left(\bar{z}\frac{d\bar{x}}{dt} - \bar{x}\frac{d\bar{z}}{dt}\right)$$
$$= B'\Omega_y - (\Sigma m\,yz)\,\Omega_z - (\Sigma m\,yx)\,\Omega_x.$$

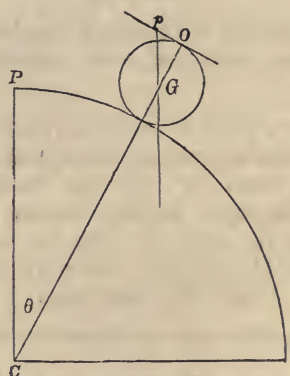
$$C\omega_z - (\Sigma m\,z'x')\,\omega_x - (\Sigma m\,z'y')\,\omega_y + \Sigma m\left(\bar{x}\frac{d\bar{y}}{dt} - \bar{y}\frac{d\bar{x}}{dt}\right)$$
$$= C'\Omega_z - (\Sigma m\,zx)\,\Omega_x - (\Sigma m\,zy)\,\Omega_y.$$

These equations determine Ω_x, Ω_y, Ω_z. It is very obvious that they may be greatly simplified by so choosing the axes that one of the two sets Ox, Oy, Oz or Gx', Gy', Gz' may be a set of principal axes.

184. Ex. *A sphere in colatitude θ is hung up by a point O in its surface in equilibrium under the action of gravity. Suddenly the rotation of the earth is stopped, it is required to determine the motion of the sphere.*

Let G be the centre of the sphere, O its point of suspension, and a its radius. Let C be the centre of the earth.

Let $\omega =$ angular velocity of the earth, then if $CG = \mu a$, the sphere is turning about an axis GP parallel to CP, the



axis of the earth, with angular velocity ω, while the centre of gravity is moving with velocity $\mu a \sin\theta \,.\, \omega$.

Let OC, Op, and the perpendicular to the plane of OC, Op be taken as the axes of x, y, z respectively, and let Ω_x, Ω_y, Ω_z be the angular velocities about them just after the rotation of the earth is stopped.

By Art. 183, the areas conserved about Ox as axis just before and just after the rotation was stopped are equal to each other.

$$\therefore\ Mk^2\,\omega\cos\theta = Mk^2\,\Omega_x,$$

where Mk^2 is the moment of inertia of the sphere about a diameter.

Again, the areas conserved about Oy are equal to each other;

$$\therefore\ Mk^2\,\omega\sin\theta + M\mu\,a^2\,\omega\sin\theta = M(k^2+a^2)\,\Omega_y.$$

Lastly, the areas conserved about Oz are equal;

$$\therefore\ O = Mk^2\,\Omega_z.$$

Solving these equations, we get

$$\Omega_y = \omega\sin\theta\,\frac{k^2+\mu a^2}{k^2+a^2}$$

$$= \omega\sin\theta\,\frac{2+5\mu}{7}.$$

But $\qquad \Omega_x = \omega\cos\theta.$

Adding together the squares of Ω_x, Ω_y, Ω_z we have

$$\Omega^2 = \omega^2\left\{\cos^2\theta + \left(\frac{2+5\mu}{7}\right)^2\sin^2\theta\right\},$$

where Ω is the angular velocity of the sphere about its instantaneous axis.

Sect. II. *Vis Viva.*

185. Let x, y, z be the co-ordinates of a particle m of a system in motion at any given instant; and let X, Y, Z be the accelerating forces acting on the particle resolved in the directions of the axes. Then, the summation being extended throughout the system, the function

$$\Sigma m\,(Xdx + Ydy + Zdz)$$

will in general be a complete differential of some quantity U. This quantity U, when it exists, is called *the force function.*

186. Prop. I. *To prove that there will be a force function first, when the forces tend to fixed centres at finite distances, and are functions of the distances from those centres;*

and secondly, when the forces are due to the mutual attractions or repulsions of the particles of the system, and are functions of the distances between the attracting or repelling particles.

Let $m\phi(r)$ be the action of any fixed centre of force on a particle m distant r, estimated in the direction in which r is measured, i.e. *from* the centre of force. Then all the forces due to the several centres of force are exactly equivalent to all their several components mX, mY, mZ. Hence, reversing the latter, there will be equilibrium. Therefore, by Virtual Velocities,

$$\Sigma m\phi(r)\,dr = \Sigma m\,(Xdx + Ydy + Zdz).$$

Thus the force function exists, and is equal to

$$\Sigma m\int\phi(r)\,dr.$$

In the same way it can be shown that there will be a force function when the forces are such as result from the attraction of the particles of the system on each other, provided the attractions are functions only of the distances between the particles.

Let $mm'\phi(r)$ be the action between two particles m, m', whose distance apart is r; and, as before, let this force be considered positive when repulsive. Then we have

$$\Sigma mm'\phi(r)\,dr = \Sigma m\,(Xdx + Ydy + Zdz),$$

and the force function is equal to

$$\Sigma mm'\int\phi(r)\,dr.$$

187. PROP. II. *If a system receive any small displacement ds parallel to a given straight line and an angular displacement $d\theta$ round that line, then the partial differential coefficients $\dfrac{dU}{ds}$ and $\dfrac{dU}{d\theta}$ represent respectively the resolved part of all the forces along the line and the moment of the forces about it.*

Since dU is the sum of the virtual moments of all the forces due to any displacement, it is independent of any particular co-ordinate axes. Let the straight line along which ds is measured be taken as the axis of z.

Taking the same notation as before,

$$dU = \Sigma m\,(Xdx + Ydy + Zdz).$$

But $dx = 0$, $dy = 0$, and $dz = ds$, hence we have

$$dU = ds\,.\,\Sigma mZ;$$

$$\therefore\ \frac{dU}{ds} = \Sigma mZ.$$

Here dU means the change produced in U by the single displacement of the system, taken as one body, parallel to any straight line, through a space ds.

Again, the moment of all the forces about the axis of z is

$$\Sigma m\,(xY - yX),$$

but by Art. 91, $dx = -y d\theta$, and $dy = x d\theta$, and $dz = 0$, hence the above moment is

$$= \Sigma m\,.\,\frac{Ydy + Xdx + Zdz}{d\theta}$$

$$= \frac{dU}{d\theta}.$$

Here dU is the change produced in U by the single rotation of the system, taken as one body, round any axis, through an angle $d\theta$.

188. Def. The *Vis Viva* of a particle is the product of its mass into the square of its velocity.

189. Prop. III. *If a system be in motion under the action of finite forces, and if the geometrical relations of the parts of the system be expressed by equations which do not contain the time explicitly, the change in the vis viva of the system in passing from any one position to any other is equal to twice the corresponding change produced in the force function.*

In determining the force function all forces may be omitted which would not appear in the equation of Virtual Velocities.

Let x, y, z be the co-ordinates of any particle m, and let X, Y, Z be the resolved parts in the directions of the axes of the impressed accelerating forces acting on the particle.

The effective forces acting on the particle m at any time t are

$$m\frac{d^2x}{dt^2},\quad m\frac{d^2y}{dt^2},\quad m\frac{d^2z}{dt^2}.$$

If the effective forces on all the particles be reversed, they will be in equilibrium with the whole group of impressed forces by Art. 28. Hence, by the principle of virtual velocities,

$$\Sigma m\left\{\left(X-\frac{d^2x}{dt^2}\right)\delta x+\left(Y-\frac{d^2y}{dt^2}\right)\delta y+\left(Z-\frac{d^2z}{dt^2}\right)\delta z\right\}=0,$$

where δx, δy, δz are any small arbitrary displacements of the particle m consistent with the geometrical relations at the time t.

Now if the geometrical relations be expressed by equations which do not contain the time explicitly, the geometrical relations which hold at the time t will hold throughout the time δt; and therefore we can take the *arbitrary* displacements δx, δy, δz to be respectively equal to the *actual* displacements

$$\frac{dx}{dt}\delta t,\quad \frac{dy}{dt}\delta t,\quad \frac{dz}{dt}\delta t$$

of the particle in the time δt.

Making this substitution, the equation becomes

$$\Sigma m\left(\frac{d^2x}{dt^2}\frac{dx}{dt}+\frac{d^2y}{dt^2}\frac{dy}{dt}+\frac{d^2z}{dt^2}\frac{dz}{dt}\right)$$
$$=\Sigma m\left(X\frac{dx}{dt}+Y\frac{dy}{dt}+Z\frac{dz}{dt}\right).$$

Integrating, we get

$$\Sigma m\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}$$
$$=C+2\Sigma m\int(Xdx+Ydy+Zdz),$$

where C is the constant to be determined by the initial conditions of motion.

Let v and v' be the velocities of the particle m at the times t and t'. Also let U, U' be the values of the force function for the system in the two positions which it has at the times t and t'. Then

$$\Sigma mv'^2 - \Sigma mv^2 = 2(U' - U).$$

190. Let a force P act on a particle which describes the elementary arc ds in the time dt. Let dp be the projection of ds on the line of action of P estimated positive in the direction in which P acts. Then Pdp is called the "*work*" done by the force P in the time dt. If a force act continuously on a particle during a time T, then the work will clearly be represented by $\int_0^T \frac{Pdp}{dt}\,dt$. Let X, Y, Z be the resolved parts in the directions of the axes of the accelerating forces which act on any particle m. Then the whole "work" done, while the system is moving from one position to another, is

$$W = \Sigma m \int (Xdx + Ydy + Zdz).$$

Let U, U' be the values of the force function of the system corresponding to the two positions of the system. Then

$$W = U' - U.$$

By the principle of vis viva we have

$$\Sigma mv'^2 - \Sigma mv^2 = 2(U' - U).$$

Therefore the change in the vis viva of a system in moving from one position to another is equal to twice the "work" done by the forces during the motion.

In every dynamic system there are three quantities, which are equal to each other. These are, briefly, the change in the vis viva, twice the change in the force function, and twice the "work" done by the forces. When the system consists only of one particle, the force function is also called the *potential*.

191. The utility of the Principle of Vis Viva depends in great measure on the fact that all the unknown reactions

of the system do not appear in the equation. All forces and reactions will not appear in the force function which would not appear in the equation of virtual velocities. These forces may be enumerated as follows:

I. Those reactions whose virtual velocities are zero.

1. Those whose line of action passes through an instantaneous axis; as *rolling friction*, but not sliding friction nor the resistance of any medium.

2. Those whose line of action is perpendicular to the direction of motion of the point of application; as the reactions of *smooth fixed surfaces*, but not those of moving surfaces.

II. Those reactions whose virtual velocities are not zero and which therefore would enter into the equation, but which disappear when joined to other reactions.

1. The reactions between particles whose distance apart remains the same; as the tensions of *inextensible strings*, but not those of elastic strings.

2. The reaction between two rigid bodies, parts of the same system, which roll on each other. It is necessary however to include both these bodies in the *same* equation of vis viva.

III. All tensions which act along inextensible strings, even though the strings are bent by passing through smooth fixed rings.

For let a string whose tension is T connect the particles m, m', and pass through a ring distant respectively r, r' from the particles. The virtual velocity is clearly

$$T\delta r + T\delta r',$$

because the tension acts along the string. But since the string is inextensible

$$\delta r + \delta r' = 0;$$

therefore the virtual velocity is zero.

192. If a system be under the action of no external forces, we have

$$X=0, \quad Y=0, \quad Z=0,$$

and hence the vis viva of the system is constant.

If, however, the mutual reactions between the particles of the system are such as would appear in the equation of virtual moments, then the vis viva of the system will not be constant. Thus, even if the solar system were not acted on by any external forces, yet its vis viva would not be constant. For the mutual attractions between the several planets are reactions between particles whose distance does not remain the same, and hence the sum of the virtual moments will not be zero. See also Art. 186.

Again, if the earth be regarded as a body rotating about an axis and slowly contracting from loss of heat in course of time, the vis viva will not be constant, for the same reason as before. The increase of angular velocity produced by this contraction can be easily found by the conservation of areas.

193. Let gravity be the only force acting on the system. Let the axis of z be vertical, then we have

$$X=0, \qquad Y=0, \qquad Z=-g.$$

Hence the equation of vis viva becomes

$$\Sigma mv'^2 - \Sigma mv^2 = -2Mg(z'-z).$$

Thus the vis viva of the system depends only on the altitude of the centre of gravity. If any horizontal plane be drawn, the vis viva of the system is the same whenever the centre of gravity passes through the plane.

194. Prop. *To determine the vis viva of a rigid body in motion.*

If a body move in any manner its vis viva at any instant is equal to the vis viva of the whole mass collected at its centre *of gravity, plus the vis viva round the centre of gravity considered as a fixed point: or*

The vis viva of a body = vis viva due to translation
+ vis viva due to rotation.

Let x, y, z be the co-ordinates of a particle whose mass is m and velocity v, and let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the co-ordinates of the centre of gravity G of the body.

Let $x = \bar{x} + x'$, $y = \bar{y} + y'$, $z = \bar{z} + z'$ .... (1).

Then by a property of the centre of gravity

$$\Sigma mx' = 0, \quad \Sigma my' = 0, \quad \Sigma mz' = 0.$$

Hence

$$\Sigma m \frac{dx'}{dt} = 0, \quad \Sigma m \frac{dy'}{dt} = 0, \quad \Sigma m \frac{dz'}{dt} = 0.$$

Now the vis viva of a body is

$$\Sigma mv^2 = \Sigma m \left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\}.$$

Substituting for x, y, z from (1), this becomes

$$\Sigma m \left\{\left(\frac{d\bar{x}}{dt}\right)^2 + \left(\frac{d\bar{y}}{dt}\right)^2 + \left(\frac{d\bar{z}}{dt}\right)^2\right\} + \Sigma m \left\{\left(\frac{dx'}{dt}\right)^2 + \left(\frac{dy'}{dt}\right)^2 + \left(\frac{dz'}{dt}\right)^2\right\}$$

$$+ 2\frac{d\bar{x}}{dt} \Sigma m \frac{dx'}{dt} + 2\frac{d\bar{y}}{dt} \Sigma m \frac{dy'}{dt} + 2\frac{d\bar{z}}{dt} \Sigma m \frac{dz'}{dt}.$$

All the terms in the last line vanish. The first term in the first line is the vis viva of the whole mass Σm, collected at the centre of gravity. The second term is the vis viva due to rotation round the centre of gravity.

This expression for the vis viva may be put into a more convenient shape.

First. *Let the motion be in two dimensions.* Let $\bar{v}$ be the velocity of the centre of gravity, $\bar{r}$, $\bar{\theta}$ its polar co-ordinates referred to any origin in the plane of motion. Let r' be the distance of any particle whose mass is m from the centre of gravity, and let v' be its velocity relatively to the centre of gravity. Let ω be the angular velocity of the whole body about the centre of gravity, and Mk^2 its moment of inertia about the same point.

The vis viva of the whole mass collected at G is $M\bar{v}^2$, which may by Differential Calculus be put into either of the forms

$$M\bar{v}^2 = M\left\{\left(\frac{d\bar{x}}{dt}\right)^2 + \left(\frac{d\bar{y}}{dt}\right)^2\right\}$$

$$= M\left\{\left(\frac{d\bar{r}}{dt}\right)^2 + \bar{r}^2\left(\frac{d\bar{\theta}}{dt}\right)^2\right\}.$$

The vis viva about G is $\Sigma mv'^2$. But since the body is turning about G, we have $v' = r'\omega$. Hence

$$\Sigma mv'^2 = \omega^2 . \Sigma mr'^2$$

$$= \omega^2 . Mk^2.$$

Hence the whole vis viva of the body is

$$\Sigma mv^2 = M\bar{v}^2 + Mk^2\omega^2.$$

If the body be turning about an instantaneous axis, whose distance from the centre of gravity is r, we have $\bar{v} = r\omega$. Hence

$$\Sigma mv^2 = M\omega^2 (r^2 + k^2)$$

$$= Mk'^2\omega^2,$$

where Mk'^2 is the moment of inertia about the instantaneous axis.

Secondly. *Let the body be in motion in space of three dimensions.*

Let $\bar{v}$ be the velocity of G; $\bar{r}$, θ, ϕ its polar co-ordinates referred to any origin. Let ω_x, ω_y, ω_z be the angular velocities of the body about any three axes at right angles meeting in G, and let A, B, C be the moments of inertia of the body about the axes. Let x', y', z' be the co-ordinates of a particle m referred to these axes.

The vis viva of the whole mass collected at G is $M\bar{v}^2$, which may be put into either of the following forms (see Differential Calculus):

$$M\bar{v}^2 = M\left\{\left(\frac{d\bar{x}}{dt}\right)^2 + \left(\frac{d\bar{y}}{dt}\right)^2 + \left(\frac{d\bar{z}}{dt}\right)^2\right\}$$

$$= M\left\{\left(\frac{d\bar{r}}{dt}\right)^2 + \bar{r}^2 \sin^2 \bar{\theta}\left(\frac{d\bar{\phi}}{dt}\right)^2 + \bar{r}^2\left(\frac{d\bar{\theta}}{dt}\right)^2\right\}.$$

The vis viva due to the motion about G is

$$\Sigma mv'^2 = \Sigma m\left\{\left(\frac{dx'}{dt}\right)^2 + \left(\frac{dy'}{dt}\right)^2 + \left(\frac{dz'}{dt}\right)^2\right\}.$$

But

$$\left.\begin{aligned} \frac{dx'}{dt} &= \omega_y z' - \omega_z y' \\ \frac{dy'}{dt} &= \omega_z x' - \omega_x z' \\ \frac{dz'}{dt} &= \omega_x y' - \omega_y x' \end{aligned}\right\}.$$

Substituting, we get, since $A = \Sigma m(y'^2 + z'^2)$,

$$B = \Sigma m(z'^2 + x'^2), \quad C = \Sigma m(x'^2 + y'^2).$$

$$\Sigma mv'^2 = A\omega_x^2 + B\omega_y^2 + C\omega_z^2$$

$$- 2(\Sigma m x'y')\,\omega_x\omega_y - 2(\Sigma m y'z')\,\omega_y\omega_z - 2(\Sigma m z'x')\,\omega_z\omega_x.$$

If the axes of co-ordinates be the principal axes at G, this reduces to

$$\Sigma mv'^2 = A\omega_x^2 + B\omega_y^2 + C\omega_z^2.$$

If the body be turning about a point O, whose position is fixed for the moment, the vis viva may be proved in the same way to be

$$\Sigma mv^2 = A'\omega_x^2 + B'\omega_y^2 + C'\omega_z^2,$$

where A', B', C' are the principal moments of inertia at the point O, and ω_x, ω_y, ω_z are the angular velocities of the body about the axes.

195. Ex. 1. *A circular wire can turn freely about a vertical diameter as a fixed axis, and a bead can slide freely along it under the action of gravity. The whole system being set in rotation about the vertical axis, find the subsequent motion.*

Let M and m be the masses of the wire and bead, ω their common angular velocity about the vertical. Let a be the radius of the wire, Mk^2 its moment of inertia about the diameter. Let the centre of the wire be the origin, and let the axis of y be measured vertically downwards. Let θ be the angle the radius drawn from the centre of the wire to the bead makes with the axis of y.

It is evident, since gravity acts vertically and since all the reactions at the fixed axis must pass through the axis, that the moment of all the forces about the vertical diameter is zero. Hence, by conservation of areas, we have

$$Mk^2\omega + ma^2 \sin^2\theta\,\omega = h.$$

And by the principle of vis viva,

$$Mk^2\omega^2 + m\left\{a^2\left(\frac{d\theta}{dt}\right)^2 + a^2\sin^2\theta\omega^2\right\} = C + 2mga\cos\theta.$$

These two equations will suffice for the determination of $\frac{d\theta}{dt}$ and ω. Solving them, we get

$$\frac{h^2}{Mk^2 + ma^2\sin^2\theta} + ma^2\left(\frac{d\theta}{dt}\right)^2 = C + 2mga\cos\theta.$$

This equation cannot be integrated, and hence θ cannot be found in terms of t. To determine the constants h and C we must recur to the initial conditions of motion. Supposing that initially $\theta=\pi$, and $\frac{d\theta}{dt}=0$ and $\omega=\alpha$, then $h=Mk^2\alpha$ and $C=2mga+Mk^2\alpha^2$.

196. Ex. 2. *Two equal and perfectly rough spheres are placed one on the top of the other in unstable equilibrium, the lower one resting on a perfectly smooth horizontal plane. A slight disturbance being given to the system, find the subsequent motion, supposing the centres of the spheres to move in one plane.*

Let C, C' be the centres of the lower and upper spheres, P their point of contact. Then by the principle of the conservation of the centre of gravity, the common centre of gravity P of the two spheres moves in a vertical straight line. Let this line be taken for the axis of y, and let the vertical plane in which the centres of the spheres move be taken for the plane of xy, and let the origin be in the fixed horizontal plane.

Let x, a be the co-ordinates of C, x', y' of C', and let θ be the acute angle CC' makes with the vertical. Let CA, $C'A'$ be those radii of the spheres which initially were in the same straight line. Then since one sphere *rolls* on the other, the angles ACP, $A'C'P$ are equal; let this angle be $=\phi$. Let F be the friction between the spheres.

Now the vis viva of the sphere $C=$ vis viva due to translation plus the vis viva due to rotation, Art. 194. The first of these is $M\left(\frac{dx}{dt}\right)^2$, and the latter is $Mk^2\left\{\frac{d(\theta-\phi)}{dt}\right\}^2$, since $\theta-\phi$ is the angle a fixed line CA in the body makes with a fixed line in space, viz. the vertical. Similarly the vis viva of the sphere C' is

$$=M\left(\frac{dx'}{dt}\right)^2+M\left(\frac{dy'}{dt}\right)^2+Mk^2\left\{\frac{d(\theta+\phi)}{dt}\right\}^2.$$

Now the only impressed force is gravity, therefore the force function is

$$-M\int g\,dy' = MC - Mgy'.$$

Hence, by the principle of vis viva, we have

$$\left(\frac{dx}{dt}\right)^2 + k^2\left\{\frac{d(\theta-\phi)}{dt}\right\}^2 + \left(\frac{dx'}{dt}\right)^2 + \left(\frac{dy'}{dt}\right)^2 + k^2\left\{\frac{d(\theta+\phi)}{dt}\right\}^2$$
$$= 2C - 2gy' \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Taking moments about the centre of gravity of each sphere, we have

$$k^2\frac{d^2(\theta-\phi)}{dt^2} = \frac{Fa}{M} \quad\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$k^2\frac{d^2(\theta+\phi)}{dt^2} = \frac{Fa}{M} \quad\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Also we have the geometrical equations

$$x = -a\sin\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$
$$x' = a\sin\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$
$$y' = a + 2a\cos\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

To solve these,

Subtracting (2) from (3),

$$\frac{d^2\phi}{dt^2} = 0;$$

$$\therefore\ \frac{d\phi}{dt} = \text{const.}$$

Since the motion starts from rest we have $\frac{d\phi}{dt} = 0$. Hence ϕ is always zero, or the two spheres descend as if they were rigidly connected.

Substituting for x, x' and y' from (4), (5), (6), equation (1) becomes

$$(k^2+a^2+a^2\sin^2\theta)\left(\frac{d\theta}{dt}\right)^2 = ga\,(\text{const.} - \cos\theta).$$

Since the initial value of θ is $\theta = 0$, then the right-hand side of this equation is clearly $= ga\,(1-\cos\theta)$. This equation gives $\dfrac{d\theta}{dt}$.

At the instant when the upper ball reaches the ground $\theta = \dfrac{\pi}{2}$, hence we then have $\left(\dfrac{d\theta}{dt}\right)^2 = \dfrac{ag}{k^2+2a^2}$.

197. Def. If a body be suspended from a fixed point O under the action of gravity, and if the angular motion of the line joining O to the centre of gravity be the same as that of a string of length l, to the extremity of which a heavy particle is attached, then l is called the *length of the simple equivalent pendulum*. This is an extension of the definition in Art. 36.

If a body turn about a fixed point under the action of gravity, there does not in general exist any such quantity as the length of a simple equivalent pendulum. Thus suppose a body, such that two of its principal moments at the centre of gravity G are equal, to be suspended from a fixed point O in the axis of unequal moment OC.

Since the moment of all the forces about a vertical through O is zero, we may apply the principle of conservation of areas with this line as axis, Art. 179.

Taking the usual notation, we have therefore

$$-A\omega_1\sin\theta + C\omega_3\cos\theta = \alpha.$$

And, by the principle of vis viva,

$$A(\omega_1^2+\omega_2^2) + C\omega_3^2 = \beta + 2Mgh\cos\theta.$$

By Euler's equations in Art. 102, we have for the motion about OC,

$$C\frac{d\omega_3}{dt} = 0.$$

Hence ω_3 is constant. Let $\beta' = \beta - C\omega_3^2$.

But since $\omega_1 = -\sin\theta \frac{d\psi}{dt}$, and $\omega_2 = \frac{d\theta}{dt}$, these equations become

$$\left.\begin{aligned} &A\sin^2\theta\frac{d\psi}{dt} + Cn\cos\theta = \alpha \\ &A\left\{\sin^2\theta\left(\frac{d\psi}{dt}\right)^2 + \left(\frac{d\theta}{dt}\right)^2\right\} = \beta + 2Mgh\cos\theta \end{aligned}\right\}.$$

To determine the arbitrary constants α and β we must have recourse to the initial values of θ and ψ. Let θ_0, ψ_0, $\frac{d\theta_0}{dt}$, $\frac{d\psi_0}{dt}$ be the initial values of θ, ψ, $\frac{d\theta}{dt}$, $\frac{d\psi}{dt}$, then the above equations become

$$\left.\begin{aligned} &\sin^2\theta\frac{d\psi}{dt} + \frac{Cn}{A}\cos\theta = \sin^2\theta_0\frac{d\psi_0}{dt} + \frac{Cn}{A}\cos\theta_0 \\ &\sin^2\theta\left(\frac{d\psi}{dt}\right)^2 + \left(\frac{d\theta}{dt}\right)^2 = \sin^2\theta_0\left(\frac{d\psi_0}{dt}\right)^2 + \left(\frac{d\theta_0}{dt}\right)^2 + 2\frac{Mgh}{A}(\cos\theta - \cos\theta_0) \end{aligned}\right\} (1).$$

These equations, when solved, give θ and ψ in terms of t, and thus determine the motion of the line OG. The corresponding equations for the motion of the simple equivalent pendulum OL are found by making $C=0$, $A=Ml^2$, and $h=l$, where l is the length of the pendulum. This gives

$$\left.\begin{aligned} &\sin^2\theta\frac{d\psi}{dt} = \sin^2\theta_0\frac{d\psi_0}{dt} \\ &\sin^2\theta\left(\frac{d\psi}{dt}\right)^2 + \left(\frac{d\theta}{dt}\right)^2 = \sin^2\theta_0\left(\frac{d\psi_0}{dt}\right)^2 + \left(\frac{d\theta_0}{dt^2}\right)^2 + 2\frac{g}{l}\cos\theta \end{aligned}\right\} \ldots\ldots (2).$$

In order that the motions of the two lines OG and OL may be the same, the two equations (1) and (2) must be the same. But this cannot be the case unless

$$Cn\cos\theta = 0;$$

i.e. unless either $n=0$, or $C=0$. Hence the body must either have no rotation about OG, or else the body must be a rod. In either case, the two sets of equations are identical if

$$\frac{2Mgh}{A}=\frac{2g}{l},$$

or

$$Mlh=A.$$

Let $A=Mk'^2$, then the length of the simple pendulum is given by

$$lh=k'^2.$$

This is the same formula which was obtained in Art. 37, where the body was supposed to move in a vertical plane.

SECT. III. *Virtual Velocities.*

198. The Principle of Virtual Velocities is of the greatest use in Statics, because it supplies us with equations sufficient for the solution of every problem, free from all the unknown reactions. In Dynamics also the same principle may be employed with advantage, and for the same reason.

PROP. *To obtain the general equations of motion of a system of rigid bodies in a form free from the unknown reactions.* Lagrange, *Mécanique Analytique.*

Let x, y, z be the co-ordinates of a particle m, and let X, Y, Z be the impressed accelerating forces acting on this particle. Then we have the equation

$$\Sigma m\left(\frac{d^2x}{dt^2}\delta x+\frac{d^2y}{dt^2}\delta y+\frac{d^2z}{dt^2}\delta z\right)$$

$$=\Sigma m\,(X\delta x+Y\delta y+Zdz)\,\ldots\ldots\ldots(1),$$

where δx, δy, δz are any small arbitrary displacements consistent with the geometrical relations, and X, Y, Z do not contain the reactions of the system.

In the following investigation, for the sake of brevity, differential coefficients with respect to t will be denoted by accents. Thus

$$x' = \frac{dx}{dt}, \quad x'' = \frac{d^2x}{dt^2}, \text{ \&c.}$$

The quantities x, y, &c. are not independent of each other, being connected together by the geometrical relations of the system. But they may all be made to depend on a certain number of independent variables whose values will determine the position of the system at any time. Let these independent variables be θ, ϕ, ψ, &c. Then x, y, &c. are functions of θ, ϕ, &c. Let $x = f(\theta, \phi, \ldots)$, $y = F(\theta, \phi, \ldots)$, &c. If the geometrical relations contain the time explicitly, then x, y, &c. will be functions of t also;

$$\therefore \delta x = A\delta\theta + B\delta\phi + \ldots$$

$$x' = A\theta' + B\phi' + \ldots$$

where A, B, &c. are written for $f'(\theta)$, $f'(\phi)$, &c. There will be similar equations for δy, y', &c.

Now $$\frac{d^2x}{dt^2}\delta x = \frac{d}{dt}(x'\delta x) - x'\frac{d\delta x}{dt} \quad \text{..................(2)}.$$

But $$\begin{aligned} x'\delta x = (A^2\theta' + AB\phi' + \ldots)\,\delta\theta \\ + (BA\theta' + B^2\phi' + \ldots)\,\delta\phi \\ + \ldots\ldots \end{aligned}$$

and $$x'^2 = A^2\theta'^2 + 2AB\theta'\phi' + B^2\phi'^2 + \ldots$$

$$\therefore x'\delta x = \frac{1}{2}\frac{d(x'^2)}{d\theta'}\delta\theta + \frac{1}{2}\frac{d(x'^2)}{d\phi'}\delta\phi + \ldots$$

$$\therefore \frac{d}{dt}(x'\delta x) = \frac{1}{2}\frac{d}{dt}\frac{d(x'^2)}{d\theta'}\delta\theta + \frac{1}{2}\frac{d(x'^2)}{d\theta'}\frac{d\delta\theta}{dt}$$

$$+ \text{ similar terms in } \phi, \psi, \text{ \&c.} \quad \ldots\ldots(3).$$

Again, since the operations d and δ are independent, we have $d\delta x = \delta dx$, and therefore

$$\begin{aligned} x' \frac{d\delta x}{dt} &= x'\delta x' \\ &= \frac{1}{2}\delta(x'^2) \\ &= \frac{1}{2}\frac{d(x'^2)}{d\theta}\delta\theta + \frac{1}{2}\frac{d(x'^2)}{d\theta'}\delta\theta' \\ &\quad + \text{similar terms in } \phi, \psi, \text{\&c.} \quad \text{.........(4).} \end{aligned}$$

Substituting from equations (3) and (4) in (2), we get

$$\frac{d^2x}{dt^2}\delta x = \left\{\frac{d}{dt}\frac{d(x'^2)}{d\theta'} - \frac{d(x'^2)}{d\theta}\right\}\frac{1}{2}\delta\theta + \begin{cases}\text{similar terms} \\ \text{in } \phi, \psi, \text{\&c.}\end{cases}$$

By similar reasoning, we have

$$\frac{d^2y}{dt^2}\delta y = \left\{\frac{d}{dt}\frac{d(y'^2)}{d\theta'} - \frac{d(y'^2)}{d\theta}\right\}\frac{1}{2}\delta\theta + \begin{cases}\text{similar terms} \\ \text{in } \phi, \psi, \text{\&c.}\end{cases}$$

$$\frac{d^2z}{dt^2}\delta z = \left\{\frac{d}{dt}\frac{d(z'^2)}{d\theta'} - \frac{d(z'^2)}{d\theta}\right\}\frac{1}{2}\delta\theta + \begin{cases}\text{similar terms} \\ \text{in } \phi, \psi, \text{\&c.}\end{cases}$$

Let T be the vis viva of the whole body, so that

$$T = \Sigma m(x'^2 + y'^2 + z'^2).$$

Then adding together the three equations above, we have

$$\Sigma m\left(\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y = \frac{d^2z}{dt^2}\delta z\right) = \left(\frac{d}{dt}\frac{dT}{d\theta'} - \frac{dT}{d\theta}\right)\frac{1}{2}\delta\theta$$
$$+ \text{similar terms in } \phi, \psi, \text{\&c.}$$

Let U be the force function of the system, then

$$\begin{aligned} \Sigma m(X\delta x + Y\delta y + Z\delta z) &= \delta U \\ &= \frac{dU}{d\theta}\delta\theta + \begin{cases}\text{similar terms} \\ \text{in } \phi, \psi, \text{\&c.}\end{cases} \end{aligned}$$

By equation (1) the two expressions on the right hand sides of these equations are equal. But since $\delta\theta$, $\delta\phi$, &c. are independent, this equality cannot exist unless the coefficients of $\delta\theta$, $\delta\phi$, &c. are separately equal.

Therefore

$$\left.\begin{aligned} \frac{d}{dt}\frac{dT}{d\theta'} - \frac{dT}{d\theta} &= 2\frac{dU}{d\theta} \\ \frac{d}{dt}\frac{dT}{d\phi'} - \frac{dT}{d\phi} &= 2\frac{dU}{d\phi} \\ \&c. &= \&c. \end{aligned}\right\} \text{..............} \quad (5).$$

These equations are of the second order, and their number is always equal to the number of independent quantities θ, ϕ, &c. to be found. Hence they are sufficient to determine the whole motion. These are the very equations we should have obtained if we had written down the ordinary equations of motion and eliminated the unknown reactions.

The equation of vis viva has been obtained in Art. 189, by another application of the principle of virtual velocities. But the equation of vis viva is of the *first* order, and gives at once a first integral of the equations. In this respect the equation of vis viva has the advantage. But on the other hand, it only supplies us with *one* equation, and if therefore the system admits of more than one independent motion, it is insufficient for the solution of the problem.

199. Prop. *To determine the oscillations of a system of bodies about a position of equilibrium.*

Let T be the vis viva of the whole system which may be found by Art. 194, and let U be the force function. Let θ, ϕ, ... be n certain small independent quantities by which the position of the system may be determined, and whose values are zero when the system is in the position of equilibrium. Also, as before, let accents denote differential coefficients with respect to t. Then we have the n equations

$$\left.\begin{aligned} \frac{d}{dt}\frac{dT}{d\theta'} - \frac{dT}{d\theta} &= 2\frac{dU}{d\theta} \\ \&c. &= \&c. \end{aligned}\right\} \text{..................} \quad (5).$$

Let x, y, z be the co-ordinates of a particle m, then

$$\left.\begin{aligned} T &= \Sigma m\,(x'^2 + y'^2 + z'^2) \\ U &= \Sigma m f(x, y, z) \end{aligned}\right\} \quad \text{.............. (6).}$$

The co-ordinates x, y, z are functions of $\theta, \phi, \ldots$ and may be expanded by Taylor's Theorem in powers of $\theta, \phi \ldots$. Assuming that the coefficients of the first powers do not all vanish, we may, since $\theta, \phi, \ldots$ are small quantities, neglect the squares and higher powers. Let then these expansions be

$$\left.\begin{aligned} x &= \alpha + \alpha_1\theta + \alpha_2\phi + \ldots \\ y &= \beta + \beta_1\theta + \beta_2\phi + \ldots \\ z &= \gamma + \gamma_1\theta + \gamma_2\phi + \ldots \end{aligned}\right\} \quad \text{..................(7),}$$

where α, β, γ, &c. are constant quantities.

Hence

$$\left.\begin{aligned} x' &= \alpha_1\theta' + \alpha_2\phi' + \ldots \\ y' &= \beta_1\theta' + \beta_2\phi' + \ldots \\ z' &= \gamma_1\theta' + \gamma_2\phi' + \ldots \end{aligned}\right\} \quad \text{..................(8).}$$

Substituting in (6), we see that T is a function of $\theta', \phi' \ldots$ and not of $\theta, \phi, \ldots$. Hence

$$\frac{dT}{d\theta} = 0, \quad \frac{dT}{d\phi} = 0, \text{ \&c.} = 0 \quad \text{..........(9),}$$

and the equations (1) reduce to

$$\left.\begin{aligned} \frac{d}{dt}\frac{dT}{d\theta'} &= 2\frac{dU}{d\theta} \\ \text{\&c.} &= \text{\&c.} \end{aligned}\right\} \quad \text{.................. (10).}$$

Let the values of T and U after substituting from (7) and (8) in the equations (6) be

$$\left.\begin{aligned} T &= A_1\theta'^2 + A_2\phi'^2 + B_1\phi'\psi' \dots \\ U - U_0 &= a_1\theta^2 + a_2\phi^2 + b_1\phi\psi + \dots \end{aligned}\right\} \dots\dots\dots\dots (11),$$

where U_0 is the value of U when the system is in the position of equilibrium, and A_1, a_1, &c. are constants.

In these expressions all terms above the second order are to be neglected.

The terms of the first order in $U - U_0$ are absent, because by the principle of virtual velocities

$$\delta U = \Sigma m\,(X\delta x + Y\delta y + Z\delta z)$$

vanishes in a position of equilibrium so far as quantities of the first order are concerned. See also Todhunter's *Statics*, Art. 263.

Substituting in equations (10), it is obvious that the n resulting equations will be linear, and of the form

$$\left.\begin{aligned} E_1\frac{d^2\theta}{dt^2} + E_2\frac{d^2\phi}{dt^2} + \dots &= e_1\theta + e_2\phi + \dots \\ F_1\frac{d^2\theta}{dt^2} + F_2\frac{d^2\phi}{dt^2} + \dots &= f_1\theta + f_2\phi + \dots \\ \&c. &= \&c. \end{aligned}\right\} \dots (12),$$

where E_1, e_1, &c. F_1, f_1, &c. are all constants.

To solve these, we must, as usual, assume

$$\theta = k\sin(\lambda t + \kappa),\quad \phi = k'\sin(\lambda t + \kappa),\ \&c.\ \dots\dots(13).$$

Substituting in the n equations (12) we shall obtain $n-1$ equations to determine the $n-1$ ratios $\frac{k'}{k}$, $\frac{k''}{k}$, &c., together with an equation of the n^{th} degree to determine the n values of λ^2.

This substitution may be effected once for all, as follows. The equations (12) become

$$\left.\begin{aligned} -\lambda^2(E_1k+E_2k'+\ldots)&=e_1k+e_2k'+\ldots\\ -\lambda^2(F_1k+F_2k'+\ldots)&=f_1k+f_2k'+\ldots\\ \&c.&=\&c.\end{aligned}\right\}\ldots\ldots(14).$$

Referring to the manner in which the equations (12) were obtained from (10) and (11), we see that

$$\left.\begin{aligned} \frac{dT}{d\theta'}&=E_1\theta'+E_2\phi'+\ldots\\ 2\frac{dU}{d\theta}&=e_1\theta+e_2\phi+\ldots\end{aligned}\right\}.$$

Hence, if we put

$$\left.\begin{aligned} T_1&=A_1k^2+A_2k'^2+B_1k'k''+\ldots\\ U_1&=a_1k^2+a_2k'^2+b_1k'k''+\ldots\end{aligned}\right\}\ldots\ldots\ldots(15),$$

which are obtained from the expressions for T and $U-U_0$, in equations (11) by writing k for θ' and θ, k' for ϕ' and ϕ; then

$$\left.\begin{aligned} \frac{dT_1}{dk}&=E_1k+E_2k'+\ldots\\ 2\frac{dU_1}{dk}&=e_1k+e_2k'+\ldots\end{aligned}\right\}.$$

Hence the equations (14) become

$$\left.\begin{aligned} \lambda^2\frac{dT_1}{dk}+2\frac{dU_1}{dk}&=0\\ \lambda^2\frac{dT_1}{dk'}+2\frac{dU_1}{dk}&=0\\ \&c.&=0\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(16).$$

In these equations the quantities k, k', &c. enter in the first power only, and the ratios $k:k':k''$: &c. may be eliminated by the usual general methods.

200. Ex. *Two rods AB, BC, are joined by a hinge at B, so that each can freely turn about B in all directions. They are suspended from a fixed point at A, and make small oscillations in one plane about the vertical. It is required to determine the motion.*

The following investigation should be compared with that in Art. 82.

Let θ, ϕ be the small angles the rods AB, BC respectively make with the vertical. Let $2a$, $2b$ be the lengths of the rods, m, m' their masses. Let the axis of x be drawn vertically downwards through A, and let x, y be the coordinates of the centre of gravity of BC. Then

$$T = m\frac{4a^2}{3}\theta'^2 + m'\left(x'^2 + y'^2 + \frac{b^2}{3}\phi'^2\right).$$

But

$$\left.\begin{array}{l} x = 2a\cos\theta + b\cos\phi, \quad y = 2a\sin\theta + b\sin\phi\,; \\ \therefore\ x' = -2a\sin\theta\,.\,\theta' - b\sin\phi\,.\,\phi',\ y' = 2a\cos\theta\,.\,\theta' + b\cos\phi\,.\,\phi' \end{array}\right\}.$$

Substituting in the expression for T, we get

$$T = \left(\frac{4}{3}ma^2 + 4m'a^2\right)\theta'^2 + \frac{4}{3}m'b^2\phi'^2 + 4m'ab\cos(\theta - \phi)\theta'\phi'.$$

Neglecting all small quantities above the second order, this reduces to

$$T = \frac{4}{3}(m + 3m')\,a^2\theta'^2 + \frac{4}{3}m'b^2\phi'^2 + 4m'ab\theta'\phi'\,;$$

$$\therefore\ T_1 = \frac{4}{3}(m + 3m')\,a^2k^2 + \frac{4}{3}m'b^2k'^2 + 4m'ab\,kk'.$$

Also, the expression for the force function U is

$$\begin{aligned} U &= mga\cos\theta + m'gx, \\ &= (m + 2m')\,ga\cos\theta + m'gb\cos\phi, \\ &= U_0 - \frac{1}{2}(m + 2m')\,ga\,.\,\theta^2 - \frac{1}{2}m'gb\,.\,\phi^2\,; \end{aligned}$$

$$\therefore\ U_1 = -\frac{1}{2}(m + 2m')\,gak^2 - \frac{1}{2}m'gbk'^2.$$

The small oscillations being represented by

$$\theta = k \sin(\lambda t + \kappa), \quad \phi = k' \sin(\lambda t + \kappa),$$

we have by (16) the equations

$$\left.\begin{aligned} &\lambda^2 \left\{\frac{8}{3}(m + 3m') a^2 k + 4m'abk'\right\} - 2(m + 2m') gak = 0 \\ &\lambda^2 \left\{\frac{8}{3} m'b^2 k' + 4m'abk\right\} - 2m'gbk' = 0 \end{aligned}\right\} \ldots(1)$$

to determine $\frac{k'}{k}$ and λ. Eliminating $\frac{k'}{k}$ we get

$$\left\{\frac{4}{3}(m + 3m') a^2\lambda^2 - (m + 2m') ga\right\} \cdot \left\{\frac{4}{3} m'b^2\lambda^2 - m'gb\right\} = 4m'^2 a^2 b^2 \lambda^4.$$

Both the values of λ^2 obtained from this equation are real and positive. Let the values of λ thus found be $\pm\lambda_1$ and $\pm\lambda_2$. Then

$$\theta = k_1 \sin(\lambda_1 t + \kappa_1) + k_2 \sin(\lambda_2 t + \kappa_2),$$

$$\phi = k_1' \sin(\lambda_1 t + \kappa_1) + k_2' \sin(\lambda_1 t + \kappa_2),$$

where the ratios $\frac{k_1'}{k_1}$, $\frac{k_2'}{k_2}$ are known from (1) when the corresponding values of λ are substituted. The four arbitrary constants k_1, k_2, κ_1, κ_2, may be found from the initial values of θ, ϕ, θ', ϕ'.

The equations of the motion of the rods when the motion is not small may be easily obtained from the expressions for T and U in a form free from all the unknown reactions.

$$\frac{dT}{d\theta'} = \left(\frac{4}{3} ma^2 + 4m'a^2\right) 2\theta' + 4m'ab \cos(\theta - \phi)\, \phi',$$

$$\frac{dT}{d\theta} = -4m'ab \sin(\theta - \phi)\, \theta'\phi',$$

$$\frac{dU}{d\theta} = -(m + 2m') ga \sin\theta.$$

Hence the equation

$$\frac{d}{dt}\frac{dT}{d\theta'} - \frac{dT}{d\theta} = 2\frac{dU}{d\theta}$$

becomes

$$\left(\frac{4}{3}ma^2 + 4m'a^2\right) 2\frac{d^2\theta}{dt^2} + 4m'ab\cos(\theta - \phi)\frac{d^2\phi}{dt^2}$$

$$+ 4m'ab\sin(\theta - \phi)\left(\frac{d\phi}{dt}\right)^2 = -2(m + 2m')ga\sin\theta,$$

and in the same way the equation

$$\frac{d}{dt}\frac{dT}{d\phi'} - \frac{dT}{d\phi} = 2\frac{dU}{d\phi}$$

becomes

$$\frac{8}{3}m'b^2\frac{d^2\phi}{dt^2} + 4m'ab\cos(\theta - \phi)\frac{d^2\theta}{dt^2}$$

$$-4m'ab\sin(\theta - \phi)\left(\frac{d\theta}{dt}\right)^2 = -2m'gb\sin\phi.$$

These equations however cannot be easily solved.

201. PROP. *To explain the principle of the co-existence of small oscillations.*

It has been proved that the motion of any system is made up of a number of simultaneous oscillations whose general type is

$$k\sin(\lambda t + \kappa).$$

Each of these motions is called *a simple oscillation*, and if the initial conditions be properly chosen, any one term will give the law of motion, and the system will make small oscillations analogous to those of a simple pendulum. Thus the general motion of a system of bodies is made up of all the simple oscillations of which it is capable. The number of such simple oscillations is not necessarily equal to the number of moveable bodies, but is equal to the number of independent motions. When the periods of all these simple oscillations are commensurable, the whole system will return to the same state in a period equal to the least common multiple of these periods.

202. Prop. *If a system of bodies be in equilibrium under the action of any forces, to determine whether the equilibrium is stable or unstable*.*

Let the system receive any small disturbance and let the type of the consequent motion be $k \sin(\lambda t + \kappa)$. Then, following the same notation as before, the equation to determine λ is found by eliminating the ratios $k : k' : k''$: &c. from the equations

$$\left.\begin{aligned} \lambda \frac{dT_1}{dk} + 2\frac{dU_1}{dk} &= 0 \\ \lambda^2 \frac{dT_1}{dk'} + 2\frac{dU_1}{dk'} &= 0 \\ \&c. &= 0 \end{aligned}\right\} \dots\dots\dots\dots\dots\dots (1).$$

Multiplying these equations respectively by k, k', &c. and adding we get, since T_1 and U_1 are homogeneous functions of the second order†,

$$\lambda^2 T_1 + 2U_1 = 0;$$

$$\therefore \lambda^2 = -\frac{2U_1}{T_1} \dots\dots\dots\dots\dots\dots\dots (2).$$

There are three different forms which the type of motion may assume according to the nature of the values of λ. If all the values of λ^2 are positive, the type of the motion is

$$k \sin(\lambda t + \kappa).$$

In this case the motion consists of a number of simultaneous small oscillations about the position of rest. The system never departs far from its position of rest, and the equilibrium is said to be stable. If any value of λ^2 be negative, the corresponding trigonometrical expression takes the form

$$A\epsilon^{at} + B\epsilon^{-at}.$$

* See Lagrange's *Mécanique Analytique;* Duhamel's *Cours de Mécanique,* Vol. II.; and Vieille's *Cours Complémentaire d'Analyse et de Mécanique Rationelle.*

† Todhunter's *Diff. Calc.* Chap. VIII. p. 108.

In this case the motion is said to be unstable, for θ, ϕ, &c. will generally become large if a sufficient time t has elapsed. Lastly, if any value of λ^2 be imaginary, the corresponding trigonometrical expression takes the form

$$(A\epsilon^{at} + B\epsilon^{-at}) \sin (\beta t + \gamma).$$

In this case the motion is oscillatory, but as the magnitude of the oscillations continually increases with the time, the motion is said to be unstable. The motion about a position of unstable equilibrium is not necessarily such as to bring the body far from its position of rest. For if the initial conditions can be so chosen that the coefficients of all the terms of the form $A\epsilon^{at}$ vanish, the motion will always be small. In such cases the equilibrium may be said to be stable for some displacements and unstable for others.

When a system is in equilibrium, we know from Statics, that the force function U is either a maximum or a minimum*. *First, let it be a minimum*, then $U - U_0$ is positive for all values of θ, ϕ,... less than certain finite limits. Now U_1 is obtained from $U - U_0$ by writing the very small quantities k, k', &c., for θ, ϕ, &c. Hence U_1 is also positive. Again, since T is the vis viva of the system, it is a function of θ', ϕ', &c., which is essentially positive for all values of θ', ϕ', &c. Hence T_1 which is obtained from T by writing k, k', &c. for θ', ϕ', &c., is also positive. Hence, by equation (2), the values of λ^2 corresponding to real values of k, k', &c. are negative. Therefore there can be no real term of the form $k \sin (\lambda t + \kappa)$. That is, *the equilibrium is unstable.*

Secondly, let U be a maximum, then $U - U_0$ is negative, and therefore by the same reasoning as before U_1 is negative. Hence by equation (2) the values of λ^2 corresponding to real values of k, k', &c. are positive. Therefore the type of the vibration is $k \sin (\lambda t + \kappa)$. When k, k', &c. and therefore also in general λ, are imaginary, the type of vibration becomes $(A\epsilon^{at} + B\epsilon^{-at}) \sin (\beta t + \gamma)$. It remains to show that there will be no terms of the latter form. By the equation of vis viva we have

$$T = 2U + C.$$

* Todhunter's *Statics*, Art. 263.

Let T', U' be the initial values of T, U, then we have

$$T+2(U_0-U)=T'+2(U_0-U').$$

Now since U_0 is the maximum value of U, the quantities U_0-U, and U_0-U' are both positive, and since the vis viva T is essentially positive, it follows that T is always less than $T'+2(U_0-U')$. Hence the vis viva of the system never exceeds a certain limit depending on the initial conditions of motion. Hence the type of motion cannot contain any such term as $(A\epsilon^{at}+B\epsilon^{-at})\sin(\beta t+\gamma)$, for then the vis viva would go on continually increasing with the time. Hence the type of vibration is $k\sin(\lambda t+\kappa)$, or *the equilibrium is stable.*

Any further discussion of the general equations of motion of a system of rigid bodies would be out of place in so elementary a work as the present. The reader is therefore referred to Lagrange's *Mécanique Analytique.*

The two following articles are taken from Pratt's *Mechanical Philosophy.*

203. Principle of Least Action. *If a system of particles move under the influence of any forces, the value of the integral $\Sigma m\int v ds$ as the system passes from one given position to another is less than if the particles had taken any other course.*

In this principle the geometrical conditions are supposed to be such that the equation of vis viva will apply.

This is called the *Principle of Least Action;* because, in general $\Sigma . m\int v ds$ is a minimum.

Let δ be the symbol of variation in the Calculus of variations: then

$$\delta(\Sigma . m\int v ds)=\Sigma . m\int\delta(v ds)=\Sigma . m\int(v\delta . ds+ds\delta v)$$

$$=\Sigma . m\int(v\delta . ds+\frac{1}{2}dt\delta . v^2).$$

Suppose the particle m rests on a curve surface, and that R is the normal pressure, α, β, γ the angles of its direction; X, Y, Z the accelerating forces acting on m, then

$$\frac{d^2x}{dt^2} = X + \frac{R}{m}\cos\alpha,\ \frac{d^2y}{dt^2} = Y + \frac{R}{m}\cos\beta,\ \frac{d^2z}{dt} = Z + \frac{R}{m}\cos\gamma.$$

Let $L=0$ be the equation to the surface; then

$$\cos\alpha = V\frac{dL}{dx},\ \ \cos\beta = V\frac{dL}{dy},\ \ \cos\gamma = V\frac{dL}{dz};$$

$$\text{where } \frac{1}{V^2} = \frac{dL^2}{dx^2} + \frac{dL^2}{dy^2} + \frac{dL^2}{dz^2}.$$

Hence $v^2 = 2\int(Xdx + Ydy + Zdz) + 2\int\frac{R}{m}VdL$,

if the particle do not rest on a surface, $R=0$; and if it do, still $dL=0$; because we suppose the motion to be such, that particles on surfaces remain on the surfaces;

$$\therefore\ v^2 = 2\int(Xdx + Ydy + Zdz) = \phi(x, y, z) + \text{const.};$$

$$\therefore\ \frac{1}{2}\delta\,.\,v^2 = X\delta x + Y\delta y + Zdz$$

$$= \frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z - \frac{R}{m}V\delta L = \frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}dz.$$

Again, $ds^2 = dx^2 + dy^2 + dz^2$,

$$\therefore\ ds\delta\,.ds = dx\delta\,.\,dx + dy\delta\,.\,dy + dz\delta\,.\,dz;$$

$$\therefore\ v\delta\,.\,ds = \frac{dx}{dt}\delta\,.\,dx + \frac{dy}{dt}\delta\,.\,dy + \frac{dz}{dt}\delta\,.\,dz.$$

Hence $\int\left(v\delta.ds + \frac{1}{2}dt\delta\,.\,v^2\right) = \frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z + \text{const.}$

and at the limits $\delta x = 0$, $\delta y = 0$, $\delta z = 0$, because the first and last positions are given;

$$\therefore\ \int\left(v\delta\,.\,ds + \frac{1}{2}dt\delta\,.\,v^2\right) = 0,$$

$$\therefore\ \delta\,(\Sigma\,.\,m\int vds) = 0,$$

and $\Sigma . m \int v ds$ is a maximum or minimum. It is evidently a minimum, because a path of an indefinite length can always be found for any particle of the system.

COR. 1. Since $ds = vdt$ we learn that $\Sigma . m \int v^2 dt$ is a minimum, or the quantity of vis viva generated or expended during any given time is a minimum.

COR. 2. If the system consist of only one particle moving on a surface, and no forces but the normal pressure act, then $\int v ds$ is a minimum: but v is a constant, therefore $\int ds$ is a minimum, or the particle will describe the shortest curve line that can be drawn on the surface between its positions at the beginning and end of the time t.

If we compare the principle of least action with the principles of the conservation of the motion of the centre of gravity, of the conservation of areas, and of vis viva, we see that this principle only serves to determine the equations of motion, and is therefore comparatively useless since these are found by much simpler means; but the other principles, which develope important properties, have the advantage of furnishing three general integrals of the equations of motion, which are in most problems the only integrals that can be found.

PROP. *To show that the calculation of the motion of a material system may be made to depend upon the integration of a single function.*

204. We shall show this by proving a new dynamical principle discovered by Sir. W. R. Hamilton and published in the *Philosophical Transactions*, 1834.

We have seen, Art. 189, that the Principle of Virtual Velocities leads us to the dynamical equation

$$\Sigma m \left\{ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \right\} = 2\Sigma . m \int \left\{ X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt} \right\} dt.$$

Now it has been shown in Art. 186 that

$$\Sigma . m \left(X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt} \right)$$

is a perfect differential coefficient with respect to t for all the forces which exist in nature; viz. forces tending to the centre of the particles of the material universe, whether fixed or moveable. Let therefore the second side $=2(U+H)$, H being independent of t: and let $2T$ be the vis viva of the system at the time t; T_0, H_0 the values of T and H when $t=0$;

$$\therefore\ T=U+H, \text{ and } T_0=U_0+H.$$

Now if the initial circumstances of the motion be varied, then H will vary, and so also will T and U: let δ be the symbol of these variations;

$$\therefore\ \delta T=\delta U+\delta H$$

$$\text{or } \Sigma\,.\,m\left\{\frac{dx}{dt}\delta\frac{dx}{dt}+\frac{dy}{dt}\delta\frac{dy}{dt}+\frac{dz}{dt}\delta\frac{dz}{dt}\right\}$$

$$=\Sigma\,.\,m\left\{\frac{d^2x}{dt^2}\delta x+\frac{d^2y}{dt^2}\delta y+\frac{d^2z}{dt^2}\delta z\right\}+\delta H;$$

$$\text{and therefore } 2\Sigma\,.\,m\left\{\frac{dx}{dt}\delta\frac{dx}{dt}+\frac{dy}{dt}\delta\frac{dy}{dt}+\frac{dz}{dt}\delta\frac{dz}{dt}\right\}$$

$$=\Sigma\,.\,m\frac{d}{dt}\left\{\frac{dx}{dt}\delta x+\frac{dy}{dt}\delta y+\frac{dz}{dt}\delta z\right\}+\delta H.$$

Now let the accumulation of the vis viva from the commencement to the termination of the time t be V;

$$\therefore\ V=\int_0^t\Sigma\,.\,m\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2dt^2.$$

Then V is a function of the initial and final co-ordinates of the material particles, and

$$\delta V=\Sigma\,.\left\{\frac{\delta V}{\delta x}\delta x+\frac{\delta V}{\delta y}\delta y+\frac{\delta V}{\delta z}\delta z+\frac{\delta V}{\delta a}\delta a+\frac{\delta V}{\delta b}\delta b+\frac{\delta V}{\delta c}\delta c\right\}$$

$$=2\int_0^t\Sigma\,.\,m\left\{\frac{dx}{dt}\delta\frac{dx}{dt}+\frac{dy}{dt}\delta\frac{dy}{dt}+\frac{dz}{dt}\delta\frac{dz}{dt}\right\}dt$$

$$+\Sigma.\left\{\frac{\delta V}{\delta a}\delta a+\frac{\delta V}{\delta b}\delta b+\frac{\delta V}{\delta c}\delta c\right\}$$

$$=\Sigma.m\left\{\frac{dx}{dt}\delta x+\frac{dy}{dt}\delta y+\frac{dz}{dt}\delta z\right\}+t\delta H+H_{,},$$

$H_{,}$ being a function of the initial co-ordinates a, b, c......

But when $t=0$, $\delta V=0$, hence

$$\delta V=\Sigma.m\left\{\frac{dx}{dt}\delta x+\frac{dy}{dt}\delta y+\frac{dz}{dt}\delta z\right\}$$

$$-\Sigma.m\left\{\frac{da}{dt}\delta a+\frac{db}{dt}\delta b+\frac{dc}{dt}\delta c\right\}+t\delta H\text{.............}(\alpha).$$

From this equation we obtain the following groups of equations; $x_1y_1z_1$ being co-ordinates to m_1......

$$\left.\begin{array}{l}\dfrac{\delta V}{\delta x_1}=m_1\dfrac{dx_1}{dt};\quad \dfrac{\delta V}{\delta x_2}=m_2\dfrac{dx_2}{dt}\dots\\[2ex] \dfrac{\delta V}{\delta y_1}=m_1\dfrac{dy_1}{dt};\quad \dfrac{\delta V}{\delta y_2}=m_2\dfrac{dy_2}{dt}\dots\\[2ex] \dfrac{\delta V}{\delta z_1}=m_1\dfrac{dz_1}{dt};\quad \dfrac{\delta V}{\delta z_2}=m_2\dfrac{dz_2}{dt}\dots\end{array}\right\}\text{.........}(A).$$

Second group,

$$\left.\begin{array}{l}\dfrac{\delta V}{\delta a_1}=-m_1\dfrac{da_1}{dt};\quad \dfrac{\delta V}{\delta a_2}=-m_2\dfrac{da_2}{dt}\dots\\[2ex] \dfrac{\delta V}{\delta b_1}=-m_1\dfrac{db_1}{dt};\quad \dfrac{\delta V}{\delta b_2}=-m_2\dfrac{db_2}{dt}\dots\\[2ex] \dfrac{\delta V}{\delta c_1}=-m_1\dfrac{dc_1}{dt};\quad \dfrac{\delta V}{\delta c_2}=-m_2\dfrac{dc_2}{dt}\dots\end{array}\right\}\text{...........}(B).$$

Lastly,

$$\frac{\delta V}{\delta H}=t\text{..........................}(C).$$

The problem is therefore reduced to finding the function V, which Sir W. R. Hamilton denominates the *characteristic function* of the motion of a system. When V is calculated, then, by eliminating H from the equations (A), (C), we shall have the $3n$ integrals of the first order of the equations of motion by simply differentiating V. And by eliminating H from the equations (B), (C), we have the $3n$ final integrals by simple differentiation.

It may be observed that V must satisfy the two following partial differential equations,

$$\frac{1}{2}\Sigma \cdot \frac{1}{m}\left\{\left(\frac{\delta V}{\delta x}\right)^2 + \left(\frac{\delta V}{\delta y}\right)^2 + \left(\frac{\delta V}{\delta z}\right)^2\right\} = U + H,$$

$$\text{and } \frac{1}{2}\Sigma \cdot \frac{1}{m}\left\{\left(\frac{\delta V}{\delta a}\right)^2 + \left(\frac{\delta V}{\delta b}\right)^2 + \left(\frac{\delta V}{\delta c}\right)^2\right\} = U_0 + H.$$

These equations furnish the principal means of discovering the form of the function V, and are of essential importance in Sir W. R. Hamilton's Theory.

The equation (α) is denominated the *law of varying action.*

205. "It has been shewn by Lagrange and others, in treating of the motion of a system, that the variation δV vanishes when the extreme co-ordinates and constant H are given (Art. 203): and they appear to have deduced from this result only the principle which is called the law of *least action:* namely, that if the particles of a system be imagined to move from a given set of initial to a given set of final positions, not as they do, nor even as they could move consistently with the general dynamical laws, or differential equations of motion, but so as not to violate any supposed geometrical connexions, nor that one dynamical relation between velocities and configuration which constitutes the law of vis viva: and if, moreover, this geometrically imaginable, but dynamically impossible motion, be made to differ infinitely little from the actual manner of motion of the system, between the given extreme positions, then the varied value of the definite integal called *action*, or the accumulated vis viva of the system in the motion thus imagined, will differ infinitely less from the actual value of that integral.

"But when this principle of least action, or," as Sir W. R. Hamilton proposed to call it, "of *stationary action*, is applied to the determination of the actual motion of a system, it serves only to form, by the rules of the Calculus of Variations, the differential equations of motion of the second order, which can always be otherwise found."

In this, then, appears the excellence of this new principle called the *law of varying action*, that we pass from an actual motion to another motion dynamically possible, by varying the extreme positions of the system and (in general) the quantity H: but more especially that it serves to express, by means of a single function, not the mere differential equations of motion, but their intermediate and their final integrals.

We hope that the slight sketch we have given of this new principle will tempt our readers to consult the original Memoirs in the Transactions of the Royal Society of London for the years 1834, 1835, from which this notice has been gathered.

EXAMPLES.

1. A uniform rod is moving on a horizontal table about one extremity, and driving before it a particle of mass equal to its own, which starts from rest indefinitely near to the fixed extremity; show that when the particle has described a distance r along the rod, its direction of motion makes with the rod an angle

$$\tan^{-1}\frac{k}{\sqrt{r^2+k^2}}.$$

2. A thin uniform smooth rod is balancing horizontally about its middle point, which is fixed; a uniform rod such as just to fit the base of the tube is placed end to end in a line with the tube, and then shot into it with such a horizontal

velocity that its middle point shall only just reach that of the tube, supposing the velocity of projection to be known, find the angular velocity of the tube and rod at the moment of the coincidence of their middle points.

Result. If m be the mass of the rod, m' that of the tube, and $2a$, $2a'$ their respective length, v the velocity of the rod's projection, ω the required angular velocity, then

$$\omega^2 = \frac{3mv^2}{ma^2 + m'a'^2}.$$

3. A fine circular tube, carrying within it a heavy particle, is set revolving about a vertical diameter. Show that the difference of the squares of the absolute velocities of the particle at any two given points of the tube equidistant from the axis is the same for all initial velocities of the particle and tube.

4. A screw of Archimedes is capable of turning freely about its axis, which is fixed in a vertical position: a heavy particle is placed at the top of the tube and runs down through it; determine the whole angular velocity communicated to the screw.

Result. Let n be the ratio of the mass of the screw to that of the particle, $\alpha =$ the angle the tangent to the screw makes with the horizon, h the height descended by the particle. Then the angular velocity generated is

$$= \sqrt{\frac{2gh \cos^2 \alpha}{(m+1).(n + \sin^2 \alpha)}}.$$

5. A cone of mass m and vertical angle 2α can move freely about it axis, and has a fine smooth groove cut along its surface so as to make a constant angle β with the generating lines of the cone. A heavy particle of mass P moves along the groove under the action of gravity, the system being initially at rest with the particle at a distance c from the vertex. Show that if θ be the angle through which the cone has

turned when the particle is at any distance r from the vertex, then

$$\frac{mk^2 + Pr^2 \sin^2 \alpha}{mk^2 + Pc^2 \sin^2 \alpha} = \epsilon^{2\theta \sin \alpha \,.\, \cot \beta},$$

k being the radius of gyration of the cone about its axis.

6. Two equal beams connected by a hinge at their centres of gravity so as to form an X are placed symmetrically on two smooth pegs in the same horizontal line, the distance between which is b. Show that, if the beams be perpendicular to each other at the commencement of the motion, the velocity of their centre of gravity when in the line joining the pegs is equal to $\sqrt{\dfrac{b^3 g}{b^2 + k^2}}$, where k is the radius of gyration of either beam about a line perpendicular to it through its centre of gravity.

7. A lamina of any form rolls on a perfectly rough straight line under the action of no forces; prove that the velocity v of the centre of gravity G is given by

$$v^2 = c^2 \frac{r^2}{r^2 + k^2},$$

where r is the distance of G from the point of contact, and k is the radius of gyration of the body about an axis through G perpendicular to its plane, and c is some constant.

8. If an elastic string, whose natural length is that of a uniform rod, be attached to the rod at both ends and suspended by the middle point, prove by means of vis viva that the rod will sink until the strings are inclined to the horizon at an angle θ, which satisfies the equation

$$\cot^3 \frac{\theta}{2} - \cot \frac{\theta}{2} - n = 0,$$

where the tension of the string, when stretched to double its length, is n times the weight.

If the string be suspended by a point, not in the middle, write down the equation of vis viva.

9. Two smooth equal beads which can slide on a wire bent into the form of an ellipse are placed at rest at the opposite extremities of any diameter. Supposing the ellipse to be freely moveable, and that the particles attract each other with a force which varies inversely as the square of the distance, determine the angular velocity of the ellipse at the moment when the beads are at the opposite extremities of the minor axis.

10. A circular wire ring, carrying a small bead, lies on a smooth horizontal table; an elastic thread the natural length of which is less than the diameter of the ring, has one end attached to the bead and the other to a point in the wire; the bead is placed initially so that the thread coincides very nearly with a diameter of the ring; find the vis viva of the system when the string has contracted to its original length.

11. A tube of given length is formed into a curve having its extremities at two fixed points in a horizontal line, a uniform chain of the same length as the tube is placed entirely within it and then slightly disturbed, determine the form of the tube that the velocity of the chain when it quits the tube may be as great as possible.

12. A beam whose mass is M is fixed at one end C, about which it can move freely in a smooth horizontal plane, and a string with a mass M' at its extremity is attached to the other end of the beam; if the whole be set in motion on the horizontal plane, so that the string shall remain constantly stretched, determine the motion.

13. A chain fixed at two points to a vertical axis revolves uniformly about it, find the differential equation of the curve which it forms by the condition that the function which expresses the total work of the forces shall be a maximum, and show how the arbitrary constants are to be determined.

14. A small insect moves along a uniform bar of mass equal to itself, and length $2a$, the extremities of which are constrained to remain on the circumference of a fixed circle, whose radius is $\frac{2a}{\sqrt{3}}$. Supposing the insect to start from the middle point of the bar, and its velocity relatively to the bar to be uniform and equal to V; prove that the bar in time t will turn through an angle

$$\frac{1}{\sqrt{3}} \tan^{-1} \frac{Vt}{a}.$$

15. A heavy circular disc is revolving in a horizontal plane about its centre which is fixed. An insect walks from the centre uniformly along a certain radius and then flies away. Determine the whole motion.

16. The extremities of a rigid rod are constrained to move on a smooth fixed wire in the form of a curve on a horizontal plane, determine the point from which a small animal must begin to move along the rod with a given relative velocity, in order that the initial angular velocity thus communicated to the rod may be the greatest possible.

17. A uniform circular disc moveable about its centre in its own plane (which is horizontal) has a fine groove in it cut along a radius, and is set rotating. A small rocket whose weight is $\frac{1}{n}$ the weight of the disc is placed at the inner extremity of the groove and discharged; and when it has left the groove, the same is done with another equal rocket, and so on. Find the angular velocity after n of these operations, and if n be indefinitely increased, find the limiting value of the same.

18. A straight tube of given length is capable of turning freely about one extremity in a horizontal plane, two equal particles are placed at different points within it at rest, an angular velocity is given to the system, determine the velocity of each particle on leaving the tube.

19. A rigid body is rotating about an axis through its centre of gravity, when a certain point of the body becomes suddenly fixed, the axis being simultaneously set free; find the equations of the new instantaneous axis; and prove that, if it be parallel to the originally fixed axis, the point must lie in the line represented by the equations

$$a^2lx + b^2my + c^2nz = 0,$$

$$(b^2 - c^2)\frac{x}{l} + (c^2 - a^2)\frac{y}{m} + (a^2 - b^2)\frac{z}{n} = 0;$$

the principal axes through the centre of gravity being taken as axes of co-ordinates, a, b, c the radii of gyration about these lines, and l, m, n the direction-cosines of the originally fixed axis referred to them.

20. A circular disc is revolving in its own plane about its centre; if a point in the circumference become fixed, find the new angular velocity.

21. Show how to deduce the equation of vis viva $T = 2U + C$ from the equations of Art. 198.

CHAPTER VIII.

ON IMPULSIVE FORCES.

SECT. I. *General Principles.*

206. IN order to understand the nature of an impulse, let us first take the simpler case of motion in two dimensions.

If a force F act on a body of mass m always in the same direction, the equation of motion of the centre of gravity is

$$\frac{dv}{dt} = \frac{F}{m},$$

where v is the velocity of the centre of gravity at the time t. Let T be the interval during which the force acts, and let v, v' be the velocities at the beginning and end of the interval. Then

$$v' - v = \frac{\int_0^T F dt}{m} \quad \text{......................} \quad (1).$$

Similarly if p be the perpendicular from the centre of gravity on the line of action of F, we have

$$\frac{d\omega}{dt} = \frac{Fp}{mk^2},$$

where ω is the angular velocity at the time t. Let ω, ω' be the angular velocities of the body at the beginning and end of the interval T. Then

$$\omega' - \omega = \frac{\int_0^T Fp\, dt}{mk^2} \quad \text{......................} \quad (2).$$

Now suppose the force F to increase without limit, while the interval T decreases without limit. Then $\int_0^T Fdt$ may have a finite limit. Let this limit be P. Then the equation (1) becomes

$$v' - v = \frac{P}{m} \qquad (3).$$

The velocity in the interval T has increased or decreased from v to v'. Supposing the velocity to have remained finite, let V be the greatest value of the velocity during this interval. Then the space described is less than VT. But in the limit this vanishes. Hence the centre of gravity has not moved during the action of the force F. It has not had time to move, but its velocity is suddenly changed from v to v'.

In the same way the angle turned through by the body in the time T is zero. Hence p will not be altered during the action of the force. The equation (2) then becomes

$$\omega' - \omega = \frac{p\int_0^T Fdt}{mk^2}$$

$$= \frac{Pp}{mk^2} \qquad (4).$$

These two equations completely determine the change in the motion of the body due to the action of the force F.

207. Such a force is called an impulse. It may be defined as the limit of a force which is infinitely great, but acts only during an infinitely short time. There are of course no such forces in nature, but there are forces which are very great, and act only during a very short time. The blow of a hammer is a force of this kind. They may be treated as if they were impulses, and the results will be more or less correct according to the magnitude of the force and the shortness of the time of action. They may also be treated as if

they were finite forces, and the displacement of the body during the time of action of the force may be found.

The quantity P may be taken as the measure of the force. An impulsive force is measured by the whole momentum generated by the impulse.

208. PROP. *In determining the effect of an impulse on a body, the effect of all finite forces which act on the body at the same time may be omitted.*

For let a finite force f act on a body at the same time as an impulsive force F. Then as before we have

$$v' - v = \frac{\int_0^T F dt}{m} + \frac{\int_0^T f dt}{m}$$

$$= \frac{P}{m} + \frac{fT}{m}.$$

But in the limit fT vanishes. Similarly the force f may be omitted in the equation of moments.

209. PROP. *To obtain the general equations of motion of a system acted on by any number of impulses at once.*

Let u, v, w, u', v', w' be the velocities of a particle of mass m parallel to the axes just before and just after the action of the impulses. Let X', Y', Z' be the resolved parts of the impulse on m parallel to the axes.

Taking the notation of Chap. II. we have the equation

$$\Sigma m \frac{d^2 x}{dt^2} = \Sigma m X,$$

or integrating

$$\Sigma m (u' - u) = \Sigma m \int_0^T X dt$$

$$= \Sigma m X' \text{.....................(1).}$$

Similarly we have the equations

$$\Sigma m\,(v' - v) = \Sigma m Y' \;\dots\dots\dots\dots\dots\dots(2),$$

$$\Sigma m\,(w' - w) = \Sigma m Z' \;\dots\dots\dots\dots\dots\dots(3).$$

Again the equation

$$\Sigma m \left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) = \Sigma m\,(xY - yX)$$

becomes on integration

$$\Sigma m \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = \Sigma m \left(x\int Ydt - y\int Xdt\right),$$

or taken between limits,

$$\Sigma m\,\{x\,(v' - v) - y\,(u' - u)\} = \Sigma m\,(xY' - yX')\dots\dots(4),$$

and the other two equations become

$$\Sigma m\,\{y\,(w' - w) - z\,(v' - v)\} = \Sigma m\,(yZ' - zY')\dots\dots(5),$$

$$\Sigma m\,\{z\,(u' - u) - x\,(w' - w)\} = \Sigma m\,(zX' - xZ')\dots\dots(6).$$

In all the following investigations it will be found convenient to use accented letters to denote the states of motion after impact which correspond to those denoted by the same letters unaccented before the action of the impulse.

210. Prop. *To prove the principle of the conservation of the centre of gravity for a system acted on by any impulses.*

This general principle may be deduced from the equations obtained in the last article. But they may also be easily obtained from the definition of an impulsive force. Let F be any finite force acting on a body during an interval of time T. The motion of the centre of gravity G is the same as if the whole force acted on the whole mass collected at G. If this be true for all values of F and T, it will be true when F is infinitely great and T infinitely small. The same reasoning will apply if there be more than one impulse acting on the body at the same moment.

It may be shown in the same way that the motion round the centre of gravity is the same as if that point was fixed.

It follows from this principle that if a system of bodies be in motion, the motion of the common centre of gravity of the whole system is not in any way affected by any explosions or impacts which may take place between the bodies.

SECT. II. *Motion of a Single Body acted on by any Impulses.*

211. PROP. *A body in motion about a fixed axis is acted on by any impulsive forces. It is required to find the pressures on the axis.*

Let the fixed axis be taken as the axis of z, let x, y, z be the co-ordinates of any particle m of the body u, v, w, u', v', w' its velocities parallel to the axes just before and just after the impulses. Let X, Y, Z be the impressed moving impulses on this particle parallel to the axes. Also let ω, ω' be the angular velocities of the body just before and after the impulses.

The impulsive pressure on the axis can be reduced to two forces acting at any two points on the axis. Let the ordinates of these points be a and a'. Let F, G, H, F', G', H' be the resolved parts of the impulses of the axis on the body at these points.

Then we have the following equations of motion:

$$\begin{aligned}\Sigma X + F + F' &= \Sigma m\,(u' - u)\\ &= -\Sigma my\,.\,(\omega' - \omega)\\ &= -M\,.\,\bar{y}\,.\,(\omega' - \omega) \;\ldots\ldots\ldots\ldots\; (1),\end{aligned}$$

since $u' = -y\omega'$ and $u = -y\omega$.

Similarly, we have

$$\begin{aligned}\Sigma Y + G + G' &= \Sigma m\,(v' - v)\\ &= +\Sigma mx\,(\omega' - \omega) \ldots\ldots\ldots\ldots\ldots\ldots (2),\\ &= +M\bar{x}\,(\omega' - \omega),\end{aligned}$$

since $v' = xw'$ and $v = xw$.

Also $\Sigma Z + H + H' = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (3).$

So also taking moments about the axes

$$\Sigma (yZ - zY) - Ga - G'a' = \Sigma m \{y (w' - w) - z (v' - v)\}$$
$$= - (\Sigma mxz) (\omega' - \omega) \ldots\ldots. (4).$$

Similarly,

$$\Sigma (zX - xZ) + Fa + F'a' = \Sigma m \{z (u' - u) - x (w' - w)\},$$
$$= - (\Sigma myz) (\omega' - \omega) \ldots\ldots. (5),$$
$$\Sigma (xY - yX) \qquad = \Sigma m (x^2 + y^2) (\omega' - \omega) \ldots\ldots (6).$$

These six equations suffice to determine ω', F, F', G, G' and the sum $H + H'$ of the two pressures along the axis.

212. These equations can be greatly simplified, *first* by taking the plane of xz to contain the centre of gravity of the body, and *secondly*, when possible, so choosing the origin that the axis of z is a principal axis in the body at that point. The equations then become

$$\Sigma X + F + F' = 0,$$
$$\Sigma Y + G + G' = M\bar{x} (\omega' - \omega),$$
$$\Sigma Z + H + H' = 0,$$
$$\Sigma (yZ - zY) - Ga - G'a' = 0,$$
$$\Sigma (zX - xZ) + Fa + F'a' = 0,$$
$$\Sigma (xY - yX) \qquad = \Sigma m (x^2 + y^2) . (\omega' - \omega).$$

213. When the fixed axis is given and the body can be so struck that there is no impulsive pressure on the axis, any point in the line of action of the force is called a *centre of percussion*.

When the line of action of the blow is given, the axis about which the body begins to turn is called the *axis of spontaneous rotation*. It obviously coincides with the position of the fixed axis in the first case.

214. Prop. *A body is capable of turning freely about a fixed axis. To determine the conditions that there shall be a centre of percussion and to find its position.*

Take the fixed axis as the axis of z, and let the plane of xz pass through the centre of gravity of the body. Let X, Y, Z be the resolved parts of the impulse, and let ξ, η, ζ be the co-ordinates of any point in its line of action. Let $M.k'^2$ be the moment of inertia of the body about the fixed axis. Then since $\bar{y}=0$, the equations of motion are

$$\left.\begin{aligned} X &= 0 \\ Y &= M\bar{x}\,(\omega'-\omega) \\ Z &= 0 \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\left.\begin{aligned} \eta Z-\zeta Y &= -(\omega'-\omega)\,\Sigma mxz \\ \zeta X-\xi Z &= -(\omega'-\omega)\,\Sigma myz \\ \xi Y-\eta X &= (\omega'-\omega)\,.\,Mk'^2 \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(2).$$

The impulsive pressures on the fixed axes are omitted because by hypothesis they do not exist.

215. From these equations we may deduce the following conditions.

First. From (1) we see that $X=0$, $Z=0$, and therefore the force must act perpendicular to the plane containing the axis and the centre of gravity.

Secondly. Substituting from (1) in the first two equations of (2) we have $\Sigma myz=0$, and $\zeta=\dfrac{\Sigma mxz}{M\bar{x}}$. Since the origin may be taken anywhere in the axis, let it be so chosen that $\Sigma mxz=0$. Then the axis of z is a principal axis at this point and $\zeta=0$. Hence the fixed axis must be a principal axis at the point where a plane passing through the line of action of the blow perpendicular to the axis cuts the axis.

Thirdly. The condition that there may be a centre of percussion, is, that the axis must be a principal axis of the body at *some* point in its length. See Art. 15.

Fourthly. Substituting from (1) in the last equation of (2), we have $\xi=\dfrac{k'^2}{\bar{x}}$. But, by Art. 37, this is the equation to determine the centre of oscillation of the body about the fixed axis treated as an axis of suspension. Hence the centres of percussion lie in the plane drawn through the centre of oscillation parallel to the fixed axis. If the fixed axis be parallel to a principal axis at the centre of gravity, the centre of oscillation coincides with a centre of percussion.

216. These results may be represented geometrically by reference to the *momental ellipsoid* at the centre of gravity. Take any central section of this ellipsoid, and let Gy, Gz be the principal diameters of the section, and GP the diametral line. Draw a plane Gxz through Gz either of these principal diameters, perpendicular to Gy the other. Then by the third condition, any straight line in the plane Gxz parallel to Gz may be taken as an axis of rotation. See Art. 15.

By the first condition the line of action of the blow must be parallel to Gy the other principal diameter, and by the fourth condition the product of its distance from the plane of yz into the distance of the axis of rotation from the same plane is equal to the square of the radius of gyration about Gz. Lastly, the line of action of the blow must pass through the diametral line GP; for let the equation to the momental ellipsoid be

$$U = Ax^2 + By^2 + Cz^2 - 2Ezx - 2Fxy - \epsilon^4 = 0.$$

The equations to the diametral line of the plane of yz are

$$\left.\begin{aligned} \frac{1}{2}\frac{dU}{dy} &= By - Fx = 0 \\ \frac{1}{2}\frac{dU}{dz} &= Cz - Ex = 0 \end{aligned}\right\}.$$

The co-ordinates of the point of application of the blow are (since $C = Mk^2$), $x = \dfrac{C}{M\bar{x}}$, $z = \dfrac{E}{M\bar{x}}$, and y may have any value. These co-ordinates evidently satisfy the above equation. Hence the line of action of the blow passes through the diametral line.

The ellipsoid of gyration at the centre of gravity may be used in a similar manner to interpret the results of Art. 215. Construct any ellipsoid confocal with the ellipsoid of gyration. Then by the second condition a normal to this ellipsoid at any point P may be an axis of rotation.

By the second condition, the line of action of the blow must lie in the tangent plane at P, see Art. 21. Draw GL a perpendicular from the centre G on the tangent plane at P, and join PL. Then by the first condition and Art. 22, the line of action of the blow is perpendicular to PL. Let it cut PL in Q, then by the fourth condition $PL \,.\, QL = k^2$, where k is the radius of gyration about GL. If the "subsidiary" ellipsoid be the ellipsoid of gyration, then $k = GL$ (Art. 12), and therefore the sphere described on PQ as diameter passes through G; hence GP, GQ are at right angles.

217. There is one case of rotation in which the results become so simple as to merit a particular discussion. This is when the fixed axis of rotation is perpendicular to a principal plane at the centre of gravity, and when the body is acted on only by a single impulse whose line of action is in the principal plane and perpendicular to the plane containing the axis of rotation and the centre of gravity.

Let the fixed axis of rotation cut the principal plane in the point S, then, as proved in Art. 19, it is a principal axis in the body at S. Let S be one of the points at which the axis of rotation is fixed, and let S' be any other, where $SS' = a$. Let the impulse R act at some point P in the straight line SG where $GP = x$. Also let $SG = h$. Then the equations of Art. 212 become

$$\left.\begin{aligned} F + F' &= 0 \\ R + G + G' &= Mh\,(\omega' - \omega) \\ G'a &= 0 \\ F'a &= 0 \\ R\,(x + h) &= M\,(k^2 + h^2)\,(\omega' - \omega) \end{aligned}\right\} .$$

Hence
$$\left.\begin{aligned} \omega' - \omega &= \frac{R(x+h)}{M(k^2+h^2)} \\ G &= R \cdot \frac{hx - k^2}{k^2+h^2} \end{aligned}\right\}.$$

The resultant impulsive action at the axis of rotation therefore passes through the point S and is perpendicular to the straight line SGP.

The impulsive action G vanishes when $x = \frac{k^2}{h}$, hence a centre of percussion always exists and coincides with the centre of oscillation.

218. When a free body turning with any angular velocity about an instantaneous axis strikes against an obstacle, it would seem that the effect of the impact is greatest if it be made at the centre of percussion; for in this case, the obstacle receives the whole motion of the body; whereas if the blow be struck in any other point, a part of the motion of the body will be employed in endeavouring to continue the motion. But this is not necessarily true.

219. Prop. *A free lamina of any form is turning in its own plane about an instantaneous centre of rotation S and impinges on an obstacle at P, situated in the straight line joining the centre of gravity G to S. To find the point P when the magnitude of the blow is a maximum.*

The following investigation will also apply if the body instead of being a lamina, be such as that described in Art. 217.

First, let the obstacle P be a fixed point.

Let $GP = x$, and let R be the force of the blow. Let $SG = h$, and let ω, ω' be the angular velocities about the centre of gravity before and after the impact. Then $h\omega$ is the linear velocity of G just before the impact; let v' be its linear velocity just after the impact.

We have the equations

$$\left.\begin{aligned} \omega' - \omega &= \frac{-Rx}{Mk^2} \\ v' - h\omega &= -\frac{R}{M} \end{aligned}\right\} \text{.....................} (1),$$

and supposing the point of impact to be reduced to rest,

$$v' + x\omega' = 0 \text{ ....} (2).$$

Substituting for ω' and v' from (1) in equation (2), we get

$$R = M\omega \, . \, k^2 \frac{x+h}{x^2+k^2} .$$

This is to be made a maximum. Equating to zero its differential coefficient with respect to x, we get

$$x^2 + 2hx - k^2 = 0 \text{ ....} (3);$$

$$\therefore \; x = -h \pm \sqrt{h^2+k^2}.$$

One of these values of x is positive and the other negative. Both these correspond to *maximum* points of percussion, but opposite in direction. Thus there is a point P with which the body strikes in front not only more forcibly than with the centre of percussion O itself, but also more forcibly than with any other point; and at the same time there is another point P' with which the body strikes with the greatest possible force, but it does so in the rear of its own translation through space*.

Let k' be the radius of gyration about the instantaneous axis of rotation, then

$$h^2 + k^2 = k'^2 \text{ and } h + x = SP;$$

$$\therefore \; SP = \pm k'.$$

Hence the two points P, P' are at equal distances from S.

* Poinsot, Sur la percussion des corps, Liouville's *Journal*, 1857; translated in the *Annals of Philosophy*, 1858.

Also if O be the centre of oscillation with respect to S as a centre of suspension, $SG \,.\, SO = k'^2$;

$$\therefore\ SP^2 = SG \,.\, SO.$$

Since GP, GP' are the roots of the quadratic equation (3),

$$\therefore\ \left.\begin{aligned} GP' - GP &= 2h \\ GP \,.\, GP' &= k^2 \end{aligned}\right\},$$

the latter equation shows that if P be made a point of suspension, P' is the corresponding centre of oscillation. It is easy to see that PP' is harmonically divided in G and O.

220. *Secondly, let the obstacle be a free particle of mass m.*

Then, besides the equations (1), we have the equation of motion of the particle m. Let V' be its velocity after impact,

$$\therefore\ V' = \frac{R}{m} \quad\text{.........................}\quad (4).$$

The point of impact in the two bodies will have after impact the same velocity, hence instead of equation (2) we have

$$V' = v' + x\omega' \quad\text{.....................}\quad (5).$$

Substituting for ω', v', V' from equations (1) and (4) in equation (5), we get

$$R = M\omega \,.\, k^2 \,.\, \frac{m\,(x+h)}{(M+m)\,k^2 + mx^2}.$$

This is to be made a maximum. Equating to zero its differential coefficient with respect to x, we get

$$x^2 + 2hx = k^2\left(1 + \frac{M}{m}\right) \quad\text{..................}\quad (6);$$

$$\therefore\ x = -h \pm \sqrt{h^2 + k^2\left(1 + \frac{M}{m}\right)}.$$

This point coincides with that found when the obstacle was fixed, only when m is infinite. To find when it coincides with the centre of oscillation, we must put $k^2 = xh$. This gives $\frac{M}{m} = \frac{x+h}{h}$, or if $l = x + h$ be the length of the simple equivalent pendulum, $\frac{M}{m} = \frac{l}{h}$.

Since $V' = \frac{R}{m}$, it is evident that when R is a maximum V' is a maximum. Hence the two points found by equation (6) might be called the centres of greatest communicated velocity.

221. There are other singular points in a moving body whose positions may be found; thus we might inquire at what point a body must impinge againt a *fixed* obstacle, that *first* the linear velocity of the centre of gravity might be a maximum, or *secondly*, that the angular velocity might be a maximum. These points, respectively, have been called by Poinsot the centres of maximum Reflexion and Conversion. Referring to equations (1) in Art. 219, we see that when v' is a maximum R is either a maximum or a minimum, and hence it may be shewn that the first point coincides with the point of greatest impact. When ω' is a maximum, we have to make

$$\omega - \frac{Rx}{Mk^2} = maximum.$$

Substituting for R, this gives

$$x^2 - 2\frac{k^2}{h}x - k^2 = 0 \text{.......................} (7).$$

If O be the centre of oscillation, we have $GO = \frac{k^2}{h}$. Let this length be represented by h'. Then the equation (7) becomes

$$x^2 - 2h'x - k^2 = 0 \text{......................} (8).$$

The roots of this equation are the same functions of h' and k that those of equation (3) are of h and k, except that the signs are opposite. Now S and O are on opposite sides of G, hence the positions of the two centres of maximum Conversion bear to O and G the same relation that the positions of the two centres of maximum Reflexion do to S and G. If the point of suspension be changed from S to O, the positions of the centres of maximum Reflexion and Conversion are interchanged.

222. PROP. *To determine the general equations of motion of a body about a fixed point under the action of given impulses.*

Let the fixed point be taken as origin, and let x, y, z be the co-ordinates of a particle m. Let u, v, w, u', v', w' be the velocities of this particle parallel to the axes before and after impact, and let X, Y, Z be the impulses on m, and L, M, N the moments of all the impulses on the body about the axes of x, y, z. And let F, G, H be the impulsive pressures of the fixed point on the body.

By D'Alembert's Principle the equations of motion are

$$\Sigma m \{x(v'-v) - y(u'-u)\} = N \quad\ldots\ldots\ldots\ldots\ldots (1),$$

and two similar equations,

$$\Sigma m (w'-w) = \Sigma Z + H \quad\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and two other similar equations.

Let w_x', w_y', w_z' be the angular velocities *generated by the impulses* about the axes of co-ordinates.

Then
$$\left.\begin{aligned} u'-u &= \omega_y' z - \omega_z' y \\ v'-v &= \omega_z' x - \omega_x' z \\ \&c. &= \&c. \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Substituting, we get

$$\Sigma m \{(x^2+y^2)\,\omega_z' - zx\omega_x' - zy\omega_y'\} = N.$$

Let A, B, C be the moments about the axes. Then we have

$$C\omega_z' - (\Sigma mzx)\,\omega_x' - (\Sigma mzy)\,\omega_y' = N,$$

similarly,

$$\left.\begin{aligned} A\omega_x' - (\Sigma mxy)\,\omega_y' - (\Sigma mxz)\,\omega_z' &= L \\ B\omega_y' - (\Sigma myz)\,\omega_z' - (\Sigma myx)\,\omega_x' &= M \end{aligned}\right\} \ldots\ldots\ldots (4).$$

These three equations will suffice to determine the values of ω_x', ω_y', ω_z'. These being added to the angular velocities before the impulse, the initial motion of the body after the impulse is found.

It is to be observed that these equations leave the axes of reference undetermined. They should be so chosen that the values of A, Σmxy, &c. may be most easily found. If the positions of the principal axes at the fixed point are known they will in general be found the most suitable.

In that case the equations reduce to the simple form

$$\left.\begin{array}{l} A\omega_x' = L \\ B\omega_y' = M \\ C\omega_z' = N \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

The values of ω_x', ω_y', ω_z' being known, we can find the pressures on the fixed point. For one of the equations (2) becomes by substitution from (3)

$$\begin{aligned} \Sigma X + F &= \Sigma m\,(\omega_y' z - \omega_z' y) \\ &= M.\,(\omega_y'\bar{z} - \omega_z'\bar{y}) \ldots\ldots\ldots\ldots\ldots (6), \end{aligned}$$

where M is the mass of the body, and $\bar{x}$, $\bar{y}$, $\bar{z}$ the co-ordinates of the centre of gravity. Similarly the other equations become

$$\Sigma Y + G = M(\omega_z'\bar{x} - \omega_x'\bar{z}),$$

$$\Sigma Z + H = M(\omega_x'\bar{y} - \omega_y'\bar{x}).$$

223. If the body be free, the motion round the centre of gravity will be the same as if that point were fixed. Hence the axes being any three straight lines meeting at the centre of gravity, the angular velocities of the body may still be found by the equations (4) or (5). The motion of the centre of gravity may be found from (2). Let $\bar{u}$, $\bar{v}$, $\bar{w}$, $\bar{u}'$, $\bar{v}'$, $\bar{w}'$ be the resolved parts of the velocities of the centre of gravity before and after the impulses, and let M be the whole mass. Then these equations become

$$\bar{w}' - \bar{w} = \frac{\Sigma Z}{M},$$

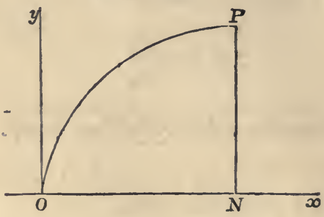
$$\bar{u}' - \bar{u} = \frac{\Sigma X}{M},$$

$$\bar{v}' - \bar{v} = \frac{\Sigma Y}{M}.$$

224. Ex. *A portion of a parabola bounded by an ordinate PN, the axis ON, and the curve OP, has its vertex O fixed. A blow P is given to it perpendicular to its plane at the other extremity of the curved boundary. Supposing it at rest before the blow, find the initial motion.*

Let the equation to the parabola be

$$y^2 = 4ax.$$



Then $\quad \Sigma m\, xz = 0, \quad \Sigma m\, yz = 0.$

Let μ be the mass of a unit of area, and M the whole mass,

$$\text{and } \Sigma m\, xy = \mu \iint xy\, dx\, dy = \mu \int x \frac{y^2}{2}\, dx \quad \begin{cases} y = 0 \\ y = y \end{cases}$$

$$= 2\mu \int ax^2\, dx = \frac{2}{3}\mu ac^3 \qquad \begin{cases} x = 0 \\ x = c \end{cases}$$

$$= M\frac{\sqrt{ac^3}}{2},$$

where $ON = C$.

Also $\quad A = \frac{1}{3}\mu \int_0^c y^3\, dx = \frac{16}{15}\mu a^{\frac{3}{2}} c^{\frac{5}{2}} = M.\frac{4ac}{5},$

$$B = \mu \int_0 x^2 y\, dx = \frac{4}{7}\mu a^{\frac{1}{2}} c^{\frac{7}{2}} = M\,\frac{3c^2}{7};$$

and the equations are

$$\left.\begin{aligned} A\omega_x - \frac{2}{3}ac^3\omega_y &= -P.2\sqrt{ac} \\ B\omega_y - \frac{2}{3}ac^3\omega_x &= Pc \\ (A+B)\,\omega_z &= 0 \end{aligned}\right\},$$

whence $\omega_x\, \omega_y$ may be easily found.

And the pressures on the fixed point may be found from the equations

$$F = 0, \quad G = 0,$$

$$H = M(\omega_x \bar{y} - \omega_y \bar{x}).$$

The axis about which the body *begins* to turn makes an angle with Ox, whose tangent is

$$\tan \theta = \frac{\omega_y}{\omega_x},$$

and the initial angular velocity $= \sqrt{\omega_x^2 + \omega_y^2}$.

But the body will not *continue* to rotate about this axis unless it be also a principal axis.

225. Prop. *A body at rest having one point fixed is acted on by an impulsive couple G, to determine the initial motion.*

Take the principal axes at the fixed point as axes of reference. The equations of motion are

$$A\omega_x = L, \quad B\omega_y = M, \quad C\omega_z = N,$$

where $L^2 + M^2 + N^2 = G^2$.

Hence the equations to the initial axis of rotation are

$$\frac{A\xi}{L} = \frac{B\eta}{M} = \frac{C\zeta}{N} \text{ .... } (1).$$

The equation to the plane of the couple is

$$L\xi + M\eta + N\zeta = 0 \text{ .... } (2).$$

Let the momental ellipsoid

$$Ax^2 + By^2 + Cz^2 = \epsilon^4$$

be constructed. Then (2) is the diametral plane of the straight line (1). Hence the initial axis of rotation is the diametral line of the plane of the impulsive couple.

It follows that the initial axis of rotation is never perpendicular to the plane of the couple, except when that plane is a principal plane of the body at the fixed point.

Let the area of the section of the ellipsoid formed by the plane of the couple G be Γ. Then

$$\frac{\pi^2\epsilon^8}{ABC} \cdot \frac{1}{\Gamma^2} = \left(\frac{L^2}{A} + \frac{M^2}{B} + \frac{N^2}{C}\right) \frac{1}{G^2}.$$

Let T be the vis viva of the body after impact, then, by Art. 194,

$$T = A\omega_x^2 + B\omega_y^2 + C\omega_z^2$$

$$= \frac{L^2}{A} + \frac{M^2}{B} + \frac{N^2}{C};$$

$$\therefore \frac{G^2}{T\Gamma^2} = \frac{ABC}{\pi^2\epsilon^8} = \text{a constant.}$$

We know by Art. 123 or 192, if the body be left to itself after the impulsive force has ceased to act, that the vis viva T will be constant throughout the subsequent motion.

226. Prop. *To show that we may take moments about the initial axis of rotation as if it were a fixed axis.*

Let l, m, n be the direction-cosines of the initial axis of rotation; let I be the moment of inertia, and Ω the angular velocity of the body about it. Let G' be the moment of the forces about the same axis. Then

$$\omega_x = l\Omega,\ \omega_y = m\Omega,\ \omega_z = n\Omega.$$

Hence the equations of motion become

$$Al\Omega = L,\ Bm\Omega = M,\ Cn\Omega = N.$$

Multiplying these by l, m, n, and adding, we get

$$(Al^2 + Bm^2 + Cn^2)\,\Omega = Ll + Mm + Cm,$$

or

$$I.\Omega = G'.$$

227. PROP. *A body at rest being acted on by any impulses, it is required to find the condition that the resulting motion may be one of rotation only.*

It has been proved that the motion of any body can always be represented by a motion of rotation about some axis, and a motion of translation in the direction of the axis. The condition that the motion may be one of rotation only is, by Art. 100,

$$\overline{u}'\omega_x' + \overline{v}'\omega_y' + \overline{w}'\omega_z' = 0.$$

But substituting for $\overline{u}'$, &c., ω_x', &c., their values in terms of the forces given in Arts. 222 and 223, this becomes

$$\frac{L\Sigma X}{A} + \frac{M\Sigma Y}{B} + \frac{N\Sigma Z}{C} = 0.$$

This condition is necessary, but not sufficient. It is also necessary that L, M, N do not all vanish.

228. PROP. *Two bodies impinge on each other, to explain the nature of the action that takes place between them.*

When two spheres of any hard material impinge on each other, they appear to separate almost immediately, and a finite change of velocity is generated in each by their mutual action. This we have seen is the characteristic of an impulsive force. Let the centres of the spheres be moving before impact in the same straight line with velocities u, v. Then after impact they will continue to move in the same straight line, and let u', v' be the velocities. Let m, m' be the masses of the spheres, R the action between them. Then we have, by Art. 209,

$$\left.\begin{aligned} u' - u &= -\frac{R}{m} \\ v' - v &= \frac{R}{m'} \end{aligned}\right\} \quad \text{.......................} \quad (1).$$

These equations are not sufficient to determine the three quantities u', v', R. To obtain a third equation we must consider what takes place during the impact.

Each of the balls will be slightly compressed by the other.

There will then obviously be two cases according as the bodies tend or do not tend to return to their original shape. In the first case they are said to be *inelastic*, in the second, *elastic*.

First, let the bodies be inelastic. While the bodies are being compressed, the motion is being propagated through their masses, but we may suppose that after a short time T, both the bodies take up new shapes and no further action takes place between them. At this moment the two bodies are moving with the same velocity. The assumptions made are, first that the change of shape and structure is so small that the effect in altering the position of the centre of gravity, and in altering the moments of inertia of the body, may be neglected, and secondly, that T is so small that the motion of the body in that time may be neglected. If for any body these assumptions are not true, the effect of impact must be deduced from the equations of the second order.

We have then just after the impact

$$u' = v' \quad\quad (2).$$

This gives

$$R = \frac{mm'}{m+m'}(u-v) \quad\quad (3),$$

whence

$$u' = \frac{mu + m'v}{m+m'} \quad\quad (4).$$

Secondly, let the bodies be elastic. Then there will be a force of restitution as well as a force of compression. Let R be the whole action between the balls, and R_0 the action that would have occurred if there had been no force of restitution. The magnitude of R must be found by experiment. This may be done by observing the values of u' and v', and thus determining the whole action by means of equations (1). These experiments were made in the first instance by Newton, and the result was that $\dfrac{R}{R_0}$ is a *constant ratio* depending on the material of the balls. This result has been confirmed by subsequent experiments.

Let the constant ratio $\frac{R}{R_0} = 1 + e$. Then e is called the common elasticity of the substances impinging. It is always less than unity. If $e = 1$, the substances are said to be perfectly elastic.

The value of e being supposed known the velocities after impact may be easily found. The action must be first calculated as if the bodies were inelastic, then the whole value of R may be found by multiplying this result by $1 + e$. This gives

$$R = \frac{mm'}{m + m'}(u - v)(1 + e),$$

whence u' and v' may be easily found by equations (1).

229. Ex. 1. *A string is wound round the circumference of a circular reel, and the free end is attached to a fixed point. The reel is then lifted up and let fall so that at the moment when the string becomes tight it is vertical, and a tangent to the reel. The whole motion being supposed to take place in one plane, determine the effect of the impulse.*

The reel in the first instance falls vertically without rotation. Let v be the velocity of the centre at the moment when the string becomes tight; v', ω' the velocity of the centre and the angular velocity just after the impulse. Let T be the impulsive tension, mk^2 the moment of inertia of the reel about its centre of gravity, a its radius.

The equations of motion are

$$v' - v = -\frac{T}{m} \quad \text{.......................} \quad (1),$$

$$\omega' = \frac{Ta}{mk^2} \quad \text{.......................} \quad (2).$$

At the moment of greatest compression the part of the reel in contact with the string has no velocity.

Hence $$v' - a\omega' = 0 \quad \text{.......................} \quad (3).$$

Substituting from (1) and (2), we have

$$T = mv \frac{k^2}{a^2 + k^2} \qquad \text{.....................} (4).$$

If the reel be a homogeneous cylinder, $k^2 = \frac{a^2}{2}$, and $T = \frac{1}{3} mv$.

In this case

$$v' = \frac{2}{3} v, \quad \text{and } \omega' = \frac{2}{3} \frac{v}{a}.$$

This will be the resulting motion if the string and reel be inelastic. If they have a common elasticity e, then we know that the correct value of T is

$$T = \frac{1}{3} mv (1 + e).$$

Substituting this in equations (1) and (2), we have

$$v' = \frac{1}{3} (2 - e) v, \quad \text{and } \omega' = \frac{2}{3} \frac{v}{a} (1 + e).$$

230. Ex. 2. *An inelastic spherical ball, moving without rotation on a smooth horizontal plane, impinges with velocity v against a rough vertical wall whose coefficient of friction is μ. The line of motion of the centre of gravity before incidence making an angle α with the normal to the wall, determine the motion after impact.*

Let v_x, v_x' be the velocities of the centre of the ball just before and just after impact resolved along the wall, v_y, v_y' the velocities resolved perpendicular to the wall in such directions that v_x and v_y are positive. Then $v_x = v \sin \alpha$, $v_y = v \cos \alpha$; let ω' be the angular velocity of the ball, and let m be its mass, and a its radius.

Let R be the normal blow, then the impulsive friction cannot be greater than μR. Let the friction be F.

The equations of motion are

$$v_x' - v_x = -\frac{F}{m} \dots\dots\dots\dots\dots\dots\dots (1),$$

$$v_y' - v_y = -\frac{R}{m} \dots\dots\dots\dots\dots\dots\dots (2),$$

$$\omega' = \frac{Fa}{mk^2} \dots\dots\dots\dots\dots\dots\dots (3).$$

At the moment of greatest compression the velocity of the point of the ball in contact with the wall must be zero. Hence we have

$$\left.\begin{aligned} v_x' - a\omega' &= 0 \\ v_y' &= 0 \end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots (2).$$

This gives

$$\left.\begin{aligned} R &= mv \cos \alpha \\ F &= m\frac{k^2}{a^2+k^2} v \sin \alpha \end{aligned}\right\} \dots\dots\dots\dots\dots\dots (3),$$

whence v_x' and ω' can be found.

This result is only true provided F is not greater than μR, or

$$\frac{k^2}{a^2+k^2} \tan \alpha \text{ not greater than } \mu,$$

$$\text{or } \frac{2}{7} \tan \alpha \text{ not greater than } \mu.$$

If α be so great that this inequality does not hold, we must have $F = \mu R$. The equations (2) now reduce to the single one

$$v_y' = 0 \dots\dots\dots\dots\dots\dots\dots\dots (4),$$

because it is no longer true that the sliding motion of the ball along the wall is destroyed. This equation gives the same value of R as before, and

$$\therefore\ F = \mu mv \cos \alpha \dots\dots\dots\dots\dots\dots (5),$$

whence v_x' and ω' can be found.

231. Ex. 3. *A cricket-ball is set rotating about a horizontal axis in the vertical plane of motion with an angular velocity* ω_1. *Supposing that when it strikes the ground the centre is moving with velocity V in a direction making an angle α with the horizon, find the subsequent motion.*

Take the normal to the ground at the point where the ball touches as the axis of z, and let the axis of x be in the plane of motion of the ball before impact. Let ω_1', ω_2', ω_3' be the angular velocities just after impact of the ball about the diameters parallel to the axes. Let u', v', w' be the resolved parts of the velocities of the centre. Let Z be the normal reaction of the ground, X, Y the frictional impulsive actions, estimated positively in the positive directions of the axes. Let m be the mass of the ball.

The equations of motion are

$$\left.\begin{aligned} \omega_1' - \omega_1 &= \frac{Ya}{mk^2} \\ \omega_2' &= -\frac{Xa}{mk^2} \\ \omega_3' &= 0 \end{aligned}\right\} \text{..................} \; (1),$$

$$\left.\begin{aligned} u' - V\cos\alpha &= \frac{X}{m} \\ v' &= \frac{Y}{m} \\ w' + V\sin\alpha &= \frac{Z}{m} \end{aligned}\right\} \text{..................} \; (2).$$

The geometrical equations are

$$\left.\begin{aligned} u' - a\omega_2' &= 0 \\ v' + a\omega_1' &= 0 \\ w' &= 0 \end{aligned}\right\} \text{......................} \; (3).$$

These hold at the moment of greatest compression. Solving these, we get

$$X = -\frac{k^2}{a^2+k^2} \cdot mV\cos\alpha,$$
$$Y = -\frac{k^2}{a^2+k^2} \cdot ma\omega_1,$$
$$Z = mV\sin\alpha.$$

Let e' be the frictional and e the normal elasticity. Then these values of X, Y, Z must be multiplied respectively by $1+e'$, $1+e'$, and $1+e$.

By substituting in (1) and (2), we get

$$\omega_1' = \omega_1 \frac{k^2 - a^2e'}{k^2+a^2}, \qquad u' = V\cos\alpha \frac{a^2 - k^2e'}{k^2+a^2},$$
$$\omega_2' = \frac{Va\cos\alpha}{k^2+a^2}(1+e'), \qquad v' = -\frac{a\omega_1 k^2}{k^2+a^2}(1+e'),$$
$$\omega_3' = 0, \qquad w' = eV\sin\alpha.$$

It appears from these equations that unless $\omega_1 = 0$ the ball will not move in the same vertical plane after impact as before. Let θ be the angle made by the two vertical planes of motion. Then $\tan\theta = \frac{v'}{u'}$, and after putting $k^2 = \frac{2}{5}a^2$, we get

$$\tan\theta = -\frac{2a\omega_1}{V\cos\alpha} \cdot \frac{1+e'}{5-2e'}.$$

232. Ex. 4. *A perfectly rough horizontal table is revolving about a vertical axis with a uniform angular velocity Ω, and a cylinder is gently placed with its plane base on the table. It is required to determine the initial motion.*

There will evidently be an impulsive friction between the base of the cylinder and the table, and since every point of the base touches the perfectly rough plane, there can be no initial rotation about a vertical axis. The cylinder will begin to turn about a tangent to some unknown point of the circumference of the base. Let this point be denoted by P. It is evident that the resultant impulsive pressure on the plane and the resultant impulsive friction must act through the point P. Since there is no initial angular velocity about the axis of the cylinder, the resultant friction must act through the centre C of the base.

Let R be the normal reaction at P, and let F be the resultant friction which acts along the radius CP.

Let ω be the initial angular velocity about the tangent at P, and let u and v be the resolved parts of the initial velocity of the centre of gravity G along and perpendicular to CP.

Let the vertical axis about which the table is turning cut the table in O, and let $OC = c$. Also let the unknown radius CP make an angle θ with CO, so that the angle $OCP = \theta$.

Let $h = CG$ and $a = CP$, and let Mk^2 be the moment of inertia of the cylinder about any horizontal axis through G.

Then the dynamical equations are

$$Mu = F,$$

$$Mv = R,$$

$$Mk^2\omega = -Fh - Ra.$$

The initial motion of the point P of the cylinder is the same as that of the point P of the table. The velocity of the point P of the cylinder is $= u - h\omega$ directed along the radius CP. The velocity of the point P of the table is $= \Omega . OP$ directed along a line perpendicular to OP. Hence the radius CP is perpendicular to OP, and therefore the axis about which the cylinder begins to turn is the tangent drawn from O to the circumference of the base. Also since $OP = c \sin\theta$, we have

$$u - h\omega = -c \sin\theta\, \Omega.$$

Again, since the point P has no vertical motion,

$$v - a\omega = 0.$$

These five equations will suffice to determine u, v, ω, F and R. Eliminating, we get

$$\frac{\omega}{\Omega} = \frac{ch \sin\theta}{k^2 + h^2 + a^2}.$$

This determines the initial angular velocity of the cylinder.

SECT. III. *The Motion of a System of Bodies acted on by any Impulses.*

233. If it be required to determine the motion of a *system* of bodies acted on by any impulses, we may proceed by writing down the equations of motion of each body separately. These equations will obviously contain the mutual reactions of the bodies that compose the system. These unknown reactions must be eliminated before we can proceed to the solution of the equations. In certain cases we may evade this elimination, and obtain an equation free from the unknown reactions in the following manner.

234. PROP. *To extend the principle of the conservation of areas to the case of a system of bodies acted on by impulses.*

Let any fixed plane be taken as the plane of xy, and any fixed point in it as origin. Let v_x, v_y, v_z, v_x', v_y', v_z' be the resolved parts of the velocities of any particle m of the system before and after the action of the impulses. Let N be the moment of the impulsive forces about the axis of z. Then we have the equation

$$\Sigma m \{x (v_y' - v_y) - y (v_x' - v_x)\} = N.$$

But the expression on the left-hand side is the difference between the area conserved by the system in two units of time before and after the action of the impulses. Hence we have, generally

$$\left.\begin{matrix}\text{area conserved}\\ \text{after any impulse}\end{matrix}\right\} - \left\{\begin{matrix}\text{area conserved}\\ \text{before the impulse}\end{matrix}\right\} = \left\{\begin{matrix}\text{moment of}\\ \text{the impulse.}\end{matrix}\right.$$

The axis about which the areas are conserved and the moment of the impulse taken is quite arbitrary, and the expressions to be used for the areas conserved are given in Art. 179.

It is obvious, that in applying this principle any internal impulses, such as an impact between two bodies of the system, may be omitted.

235. Ex. *Four equal rods, each of length 2a and mass M, are freely jointed and laid on a smooth horizontal table in the form of a square. A blow F is then struck at one corner in the direction of one of the sides. Prove that the sum of the initial angular velocities of the rod is* $\frac{3}{8}\frac{F}{Ma}$.

Let AB, BC, CD, DA be the four rods taken in order, and let G be the centre of gravity. The velocity communicated to the centre of gravity will be $\frac{F}{4M}$. Supposing this velocity applied to every particle of the system in an opposite direction, the centre of gravity will remain at rest and the initial motion will be that of twisting.

Let the blow F act at the corner B in the direction BC. Let ω be the resulting angular velocity of AB, CD; ω' that of BC, DA. Let E and H be the middle points of the rods AB, BC. Then by the proposition, since the system starts from rest,

$$\left.\begin{array}{c}\text{area conserved}\\ \text{about } G\end{array}\right\} = Fa \text{ .... (1).}$$

Now the area conserved by any body is equal to the area conserved by the centre of gravity plus the area conserved round the centre of gravity, Art. 175. The area conserved by the centre of gravity E of the rod AB about $G = Ma^2\omega'$, that conserved round $E = Mk^2\omega$, Art. 179. Hence the whole area conserved by AB is $Ma^2\omega' + Mk^2\omega$. Similarly that conserved by BC is $Ma^2\omega + Mk^2\omega'$. Hence taking the whole four rods, the equation (1) becomes

$$2M(k^2 + a^2)(\omega + \omega') = Fa;$$

$$\therefore \omega + \omega' = \frac{3}{8}\frac{F}{Ma}.$$

The principle of conservation of areas gives only one equation, and therefore cannot determine both ω and ω'. To find these we must have recourse to the equations of motion of each rod taken separate from the rest of the system, introducing into each equation the unknown reactions at the hinges; see Arts. 230, 231.

After the impulse F has ceased, the principle of conservation of areas will still hold with G for the pole of areas, Art. 178. Hence the sum of the two angular velocities will be constant during the whole of the subsequent motion.

236. PROP. *To determine the change in the vis viva of a moving system produced by any collisions between the bodies or by any explosions.*

Let $v_x, v_y, v_z, v_x', v_y', v_z'$ be the resolved parts of the velocities of any particle m of the system before and after the impulse.

Then by D'Alembert's Principle the momenta

$$m(v_x'-v_x),\ \ m(v_y'-v_y),\ \ m(v_z'-v_z),$$

being reversed and taken throughout the whole system, are in equilibrium with the forces of the impulse. But these last are themselves in equilibrium. Hence the former set are also in equilibrium. Therefore by Virtual Velocities,

$$\Sigma m\{(v_x'-v_x)\,\delta x+(v_y'-v_y)\,\delta y+(v_z'-v_z)\,\delta z\}=0,$$

where δx, δy, δz are any small arbitrary displacements of the particles impinging on each other, which are consistent with the geometrical conditions of the system during the time of action of the impulse.

During the impact, it is one geometrical condition that the particles impinging on each other have no velocity of separation normal to the common surface of the bodies of which they form a part.

First. Let the bodies be devoid of elasticity. Then the above geometrical condition will hold just *after* the moment of greatest compression as well as during the impact. Hence we can put

$$\delta x=v'_x\delta t,\ \ \delta y=v'_y\delta t,\ \ \delta z=v'_z\delta t.$$

The equation now becomes

$$\Sigma m \{(v'_x - v_x)\, v'_x + (v'_y - v_y)\, v'_y + (v'_z - v_z)\} = 0\,;$$

$$\therefore\ \Sigma m\,(v'^2_x + v'^2_y + v'^2_z) = \Sigma m\,(v_x v'_x + v_y v'_y + v_z v'_z).$$

This may be put into the form

$$\Sigma m\,(v'^2_x + v'^2_y + v'^2_z) - \Sigma m\,(v^2_x + v^2_y + v^2_z)$$
$$= -\,\Sigma m\,\{(v'_x - v_x)^2 + (v'_y - v_y)^2 + (v'_z - v_z)^2\}.$$

Therefore in the impact of inelastic bodies vis viva is always lost.

Secondly. Let an explosion take place in any body of the system. Then the geometrical equation above spoken of will hold just before the impulse begins as well as during the explosion, but it will not hold after the particles of the body have separated. Hence we must now put

$$\delta x = v_x \delta t, \quad \delta y = v_y \delta t, \quad \delta z = v_z \delta t.$$

As before, we have

$$\Sigma m\,(v_x v'_x + v_y v'_y + v_z v'_z) = \Sigma m\,(v_x^2 + v_y^2 + v_z^2),$$

and

$$\Sigma m\,(v'^2_x + v'^2_y + v'^2_z) - \Sigma m\,(v^2_x + v^2_y + v^2_z)$$
$$= +\,\Sigma m\,\{(v'_x - v_x)^2 + (v'_y - v_y)^2 + (v'_z - v_z)^2\}.$$

Therefore in cases of explosion vis viva is always gained.

Thirdly. Let the particles of the system be perfectly elastic. Then the whole action consists of two parts, a force of compression as if the particles were inelastic, and a force of restitution of the nature of an explosion. The circumstances of these two forces are exactly equal and opposite to each other. Hence by examining these two expressions it is easy to see that the vis viva lost in the compression is exactly balanced by the vis viva gained in the restitution.

EXAMPLES.

1. Three equal rods placed in a straight line are jointed by hinges to one another, they move with a velocity v perpendicular to their lengths; if the middle point of the middle one become suddenly fixed, show that the extremities of the other two will meet in a time $\frac{4\pi a}{9v}$, a being the length of each rod.

2. AB, BC are two equal uniform rods loosely jointed at B, and moving with the same velocity in a direction perpendicular to their length; if the end A be suddenly fixed, show that the initial velocity of AB is three times that of BC. Also show that in the subsequent motion of the rods, the greatest angle between them equals $\cos^{-1}\frac{2}{3}$, and that when they are next in a straight line, the angular velocity of BC is nine times that of AB.

3. Two equal rods of the same material are connected by a free joint, and placed in one straight line on a smooth horizontal table; one of them is struck perpendicularly to its length at its extremity remote from the other rod. Prove that the linear velocity communicated to its centre of gravity is $\frac{1}{4}$th greater than that which would have been communicated to it by a similar blow when free.

In the subsequent motion, prove that the rods will approach so as to form an angle $\cos^{-1}\frac{1}{3}$, but not nearer.

4. Three equal heavy uniform beams jointed together are laid in the same right line on a smooth table, and a given horizontal impulse is applied at the middle point of the centre beam in a direction perpendicular to its length; determine the instantaneous impulse on each of the other beams and the subsequent motion of the system.

5. Three beams of like substance, jointed together so as to form one beam, are laid on a smooth horizontal table. The two extreme beams are equal in length and one of them receives a blow at its free extremity in a direction perpendicular to its length. Determine the length of the middle beam in order that the greatest possible angular velocity may be given to the third.

Result. The length of the middle beam must be to either of the outer beams as $\sqrt{3} : 2$.

6. Two equal uniform rods AB, BC, loosely jointed together at B, are laid on a smooth horizontal table, so that $\angle ABC = \alpha$. A blow is struck at A perpendicular to AB; determine the direction and magnitude of the impulse at B, and show that the initial motion of A will be along BA if $\tan \alpha = \frac{1}{\sqrt{2}}$.

7. Two rough rods A, B are placed parallel to each other and in the same horizontal plane. Another rough rod C is laid across them at right angles, its centre of gravity being half way between them. If C be raised through any angle α and let fall, determine the conditions that it may oscillate, and show that if its length be equal to twice the distance between A and B, the angle θ through which it will rise in the n^{th} oscillation is given by the equation

$$\sin \theta = \left(\frac{1}{7}\right)^n . \sin \alpha.$$

8. A rod moveable in a vertical plane about a hinge at its upper end has a given uniform rod attached to its lower end by a hinge about which it can turn freely in the same vertical plane as the upper rod; at what point must the lower rod be struck horizontally in that same vertical plane that the upper rod may initially be unaffected by the blow?

9. A uniform beam is balanced about a horizontal axis through its centre of gravity, and a perfectly elastic ball is let

fall from a height h on one extremity; determine the motion of the beam and ball.

Result. Let M, m be the masses of beam and ball, $2a$ = length of beam, V, V' the velocities of ball at the moments before and after impact, ω' the angular velocity of the beam. Then

$$\omega' = \frac{6mV}{(M+3m)a}, \quad V' = V \cdot \frac{3m - M}{3m + M}.$$

10. A free lamina of any form is turning in its own plane about an instantaneous centre of rotation S and impinges on a fixed obstacle P, situated in the straight line joining the centre of gravity G to S. Find the position of P, *first*, that the centre of gravity may be reduced to rest, *secondly*, that its velocity after impact may be the same as before but reversed in direction.

Result. In the first case, P coincides either with G or with the centre of oscillation. In the second case the points $x = GP$ are found from the equation

$$x^2 - \frac{h}{2}x + \frac{k^2}{2} = 0,$$

where $SG = h$.

11. Two perfectly rough circles are revolving with different angular velocities in the same plane, and their circumferences are brought together so as to touch each other. The centre of one being fixed, determine the motion.

12. A series of equal cylinders are absolutely fixed, the axes of each being in the same horizontal plane, and each cylinder touching two others along a generating line. A heavy inelastic sphere of the same radius is passing over them in a direction perpendicular to their axes, and remaining in contact with them always. If the sphere be perfectly smooth, show that it will lose one-half of its velocity at each impact, if perfectly rough, three-fourths. Also in the first case show that it may surmount one cylinder after an impact but not more, in the second that it cannot surmount one.

13. An inelastic sphere sliding along a smooth horizontal plane impinges upon a fixed rough point; determine the condition that it may just roll over the point.

14. One half the inner surface of a fixed hemispherical bowl is smooth, and the other rough: a solid sphere slides down the smooth part of the bowl starting from rest at the horizontal rim, and at the bottom comes in contact with and rolls up the rough part of the surface. Find the change of vis viva of the sphere at the bottom of the bowl, and show that if θ be the angle which the line joining the centres of the sphere and bowl makes with the vertical when the sphere begins to descend the rough surfaces, $\cos\theta = \frac{2}{7}$.

15. A ball spinning about a vertical axis moves on a smooth table and impinges directly on a perfectly rough vertical cushion; show that the vis viva of the ball is diminished in the ratio

$$10e^2 + 14\tan^2\theta : 10 + 49\tan^2\theta,$$

where e is the elasticity of the ball and θ the angle of reflexion.

16. A lamina of any form lying on a smooth horizontal plane, is struck by a horizontal blow; determine the point about which it will begin to turn, and prove that if c, c' be the distances from the centre of gravity of the body of this point and of the direction of the blow respectively, $cc' = k^2$, where k is the radius of gyration of the lamina about the vertical line through its centre of gravity.

17. A number of equal discs are placed nearly touching each other and having their centres in the same straight line upon a smooth horizontal table, and the last has upon its centre a perfectly rough ball. If the first disc be made to impinge upon the second with a given velocity, find the time which will elapse before the ball rolls upon the table.

If the ball be removed, and if the last of the discs be not perfectly homogeneous, that is, if its centre of gravity do not coincide with its centre of figure, and if the circumferences of the discs be perfectly rough, determine the nature of the instantaneous motion, the discs being made to impinge upon each other as before.

18. A perfectly rough right prism whose section is a square is placed with its axis horizontal upon a board of equal mass lying on a smooth horizontal table. A vertical plane containing the centre of gravity of the two is perpendicular to the axis of the prism, a horizontal blow in this plane applied to the board communicates motion to the system; show that the prism will topple over if the momentum of the blow be greater than that acquired by the system falling from a height $\frac{13}{12}\tan\frac{\pi}{8}a$, where a is a side of the square.

19. A free plane lamina receives a single blow perpendicular to its plane; show that (i) if the locus of points where the blow may have been applied be a straight line, the spontaneous axis will pass through a determinate point; (ii) if the locus be a circle (centre C), the spontaneous axis will be a tangent to an ellipse whose axes are in the direction of the principal axes at C in the plane of the lamina.

20. A free oblate spheroid at rest, whose equatorial and polar axes are a and c, is struck by a blow perpendicular to its axis at any point in a plane parallel to and at a distance $\frac{a^2+c^2}{ae\sqrt{2}}$ from the equator; prove that there exists an instantaneous axis which meets the polar axis in a point P, and which, if P become fixed, will be an axis of permanent rotation.

21. A square is moving freely about a diagonal with angular velocity ω, when one of the angular points not in that diagonal becomes fixed; determine the impulsive pressure on the fixed point, and show that the instantaneous angular velocity will be $\frac{\omega}{7}$.

22. A smooth cone bounded by planes parallel to the axis, and equidistant from the vertex receives a blow at a given point. Determine the axis about which it begins to rotate, the base being an ellipse.

23. An inelastic sphere on a rough horizontal plane receives a blow which does not cause it to leave the plane. Prove that in general it will first describe a portion of a parabola, and afterwards move in a right line.

24. A uniform rough sphere of radius a, rotating with uniform angular velocity Ω about an axis through its centre, is brought into contact with another uniform rough sphere of equal size and mass whose centre is fixed rotating with equal angular velocity about an axis at right angles to the former, the line joining the centres of the spheres being perpendicular to both axes of rotation. Prove that immediately after impact, the centre of the former sphere will move in a direction equally inclined to the axes of rotation before impact, with a velocity $\frac{\sqrt{2}}{6} a\Omega$, and that each sphere will rotate with an angular velocity $\frac{\sqrt{74}}{12} . \Omega$ about an axis inclined to its former axis of rotation at an angle $\tan^{-1} \frac{5}{7}$.

25. A rigid body moves about a fixed point O and is struck by a couple whose components about the principal axes at O are L, M, N. Prove that if a second point in the line whose direction-cosines are l, m, n with respect to the principal axes be also fixed, the vis viva of the motion generated will be

$$\frac{(Ll + Mm + Nn)^2}{Al^2 + Bm^2 + Cn^2},$$

and show that this is a maximum when the line l, m, n is the instantaneous axis of rotation through O, when O only is fixed.

26. A string without weight is coiled round a rough horizontal cylinder, of which the mass is M and radius a, and which is capable of turning round its axis. To the free extremity of the string is attached a chain of which the mass is m and the length l; if the chain be gathered close up and then let go, prove that if θ be the angle through which the cylinder has turned after a time t before the chain is fully stretched,

$$Ma\theta = \frac{m}{l}\left(\frac{gt^2}{2} - a\theta\right)^2.$$

CHAPTER IX.

MISCELLANEOUS EXAMPLES.

1. A POINT moves in a plane lamina so that a tangent to its path bisects the angle between the principal axes at that point. Find its path.

Result. An ellipse or hyperbola whose centre is at the centre of gravity of the lamina.

2. If each element of the area of a triangle ABC be multiplied by the n^{th} power of its distance from a straight line passing through one angle A, then the sum of the products is

$$\frac{2}{(m+1)(m+2)} \frac{\beta^{n+1} - \gamma^{n+1}}{\beta - \gamma} . A,$$

where β and γ are the distances of the angular points B, C from the straight line through the angular point A, and A is the area of the triangle.

3. A body of any form can turn freely about one of the principal axes at the centre of gravity as a fixed axis. To determine the moment of the attraction of a very distant centre of force about that axis.

Let the centre of gravity G be the origin, and the principal axes at G the axes of co-ordinates. Let the fixed axis be the axis of y. Let x', y', z' be the co-ordinates of the centre of force S, and let $\phi(r)$ be the attraction on a unit of mass at a distance r. Let $SG = \rho$. Let x, y, z be the co-ordinates of any particle m of the body, and let r be the distance of S from m. Then the moment of the attraction about OY is

$$M = \Sigma m \frac{\phi(r)}{r} . \{z(x'-x) - x(z'-z)\}$$

$$= \Sigma m \frac{\phi(r)}{r} (x'z - xz').$$

Now $$r = \sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}$$

$$= \rho - \frac{xx' + yy' + zz'}{\rho},$$

the terms depending on the squares of small quantities being neglected;

$$\therefore \frac{\phi(r)}{r} = \frac{\phi(\rho)}{\rho} - \frac{d}{d\rho} \frac{\phi(\rho)}{\rho} . \frac{xx' + yy' + zz'}{\rho}.$$

Let $$f = \frac{\phi(\rho)}{\rho}, \quad f' = \frac{df}{d\rho}.$$

Then the moment of the force is

$$= \Sigma m \left(f - f' \frac{xx' + yy' + zz'}{\rho}\right)(x'z - xz')$$

$$= fx'\Sigma mz - fz'\Sigma mx$$

$$+ \frac{f'}{\rho} . \{(z'^2 - x'^2)\Sigma mzx + y'z'\Sigma mxy - x'y'\Sigma myz\}$$

$$+ \frac{f'}{\rho} x'z'\Sigma m(x^2 - z^2).$$

But $\Sigma mx = 0$, $\Sigma mz = 0$, and since the axes of co-ordinates are principal axes at G, $\Sigma mxy = 0$, $\Sigma myz = 0$, $\Sigma mzx = 0$. Hence the expression for the moment becomes

$$M = \frac{f'}{\rho} x'z'\Sigma m(x^2 - z^2).$$

Let A, B, C be the moments of inertia about the axes, then we have

$$M = \frac{1}{\rho} \frac{d}{d\rho} \frac{\phi(\rho)}{\rho} . (C - A) . x'z'.$$

Let θ be the angle the plane passing through S and the axis of y makes with the plane of yz; then the moment tending to turn the body from the plane of yz is

$$M = \frac{\rho^2 - y'^2}{\rho} \frac{d}{d\rho} \frac{\phi(\rho)}{\rho} \,.\, (C - A) \sin\theta \cos\theta.$$

4. An ellipsoid can turn freely about one of its principal diameters as a fixed axis, and the particles of the body are acted on by the attraction of a distant centre of force situated nearly in the principal plane perpendicular to the fixed axis. If $r\phi(r)$ be the attraction at a distance r on a unit of mass, prove that the time of a small oscillation is

$$2\pi\sqrt{\frac{B}{C-A} \cdot \frac{1}{r\phi'(r)}},$$

where B is the moment of inertia of the ellipsoid about the fixed axis; A, C the moments of inertia about the other two axes; and r is the distance of the centre of force from the centre of the ellipsoid.

5. The centre of gravity of a disc is constrained to describe an orbit which is very nearly circular about a centre of force O in its own plane. Supposing the force to vary inversely as the square of the distance, determine the angular motion of the disc.

Let GA, GC be the principal axes at G, the centre of gravity of the disc, and GB the axis perpendicular to the plane about which the disc rotates. Let A, B, C be the moments of inertia about GA, GB, GC respectively, and let C be greater than A. Also let $B = Mk^2$.

Let O be the centre of force, and let Ox be the position of OG at the time $t = 0$. Let us suppose that the disc turns on its axis in the same direction that the centre of gravity describes its circular orbit about O; and let the angle $xOG = \phi$ and the angle $OGA = \theta$, the angles being measured in the directions of rotation. Let n be the angular velocity of G about O, which will be constant throughout the motion, then $\phi = nt$.

Let the moment round G due to the attraction of O on the disc, and tending to turn it round in the direction opposite to the rotation, be $Mp^2 \sin\theta\cos\theta$.

Then by example (4)

$$Mp^2 = \frac{3\mu}{\rho^3}\,.\,(C - A),$$

where $\rho = OG$ and $\frac{\mu}{\rho^2}$ is the attraction of O at a distance ρ.

Then the equation of motion is

$$Mk^2\,\frac{d^2(\theta+\phi)}{dt^2} = -Mp^2 \sin\theta\cos\theta\,;$$

$$\therefore\ \frac{d^2\theta}{dt^2} = -\frac{p^2}{2k^2}\,.\,\sin 2\theta \ \ldots\ldots\ldots\ldots\ldots (1)\,;$$

$$\therefore\ \left(\frac{d\theta}{dt}\right)^2 = C + \frac{p^2}{2k^2}\cos 2\theta.$$

To simplify the constants let us suppose that at some instant during the motion the axis GA pointed towards the centre of force, and let the time be measured from this epoch. Let the initial value of $\frac{d\theta}{dt}$ be α. Then when $\theta = 0$, $\frac{d\theta}{dt} = \alpha$;

$$\therefore\ \alpha^2 = C + \frac{p^2}{2k^2}\,;$$

$$\therefore\ \alpha^2 - \left(\frac{d\theta}{dt}\right)^2 = \frac{p^2}{k^2}\sin^2\theta\,;$$

$$\therefore\ \alpha^2 - \frac{p^2}{k^2}\sin^2\theta = \left(\frac{d\theta}{dt}\right)^2 \ \ldots\ldots\ldots\ldots\ldots (2).$$

Hence, throughout the motion,

$$\sin\theta < \frac{k\alpha}{p}.$$

Let n' be the angular velocity of the disc about G at the time $t=0$. Then

$$\alpha = n' - n.$$

If $n'=n$, $\alpha=0$, and therefore $\theta=0$. Hence if the axis GA originally pointed towards O, it will continue to point towards O throughout the motion.

If n' be very nearly equal to n, α is very small, and therefore θ is confined between very narrow limits. Hence if the axis GA originally pointed nearly towards O, then the angular velocity of the disc would become equal to n, and the disc would move so that GA the axis of least moment would very nearly point towards O throughout the motion. In this case θ is very small and the motion may be found from the equation

$$\frac{d^2\theta}{dt^2} + \frac{p^2}{k^2}\theta = 0,$$

which is obtained by neglecting the squares and higher powers of θ in equation (1). Hence the time of a small oscillation is $\dfrac{2\pi k}{p}$.

This will explain why the moon always turns the same face towards the earth, and why the angular velocity about its axis always participates in the secular changes in the moon's mean motion.

If n' be not nearly equal to n so that $\dfrac{k\alpha}{p}$ is equal to or greater than unity, there are no limits to the value of θ. Suppose $\dfrac{k\alpha}{p}=1$, then the equation of motion becomes

$$\alpha^2(\cos^2\theta) = \left(\frac{d\theta}{dt}\right)^2;$$

$$\therefore\ \frac{d\theta}{dt} = \pm\alpha\cos\theta;$$

$$\therefore\ \sin\theta = \frac{\epsilon^{\pm 2\alpha t}-1}{\epsilon^{\pm 2\alpha t}+1},$$

the constant being determined from the condition that θ vanishes when $t=0$. Hence as t increases, $\sin\theta$ approaches

± 1, or the disc tends to take up that position in which the axis of C always points to the centre of force.

If the disc were placed initially with its axis of greatest moment, viz. GC, pointing towards O, then p^2 would be negative. If θ now represent the angle OGC, and if $-p'^2$ be written for p^2, it will be evident from equation (2) that the motion will not necessarily be such as to make GC always point to O.

6. The point of support of a simple pendulum has a small horizontal oscillatory motion represented by $x = a \sin nt$. Determine the effect on the small oscillations of the pendulum, and show that if $ln^2 = g$, where l is the length of the simple pendulum, the vibrations of the pendulum will become large. Also determine the effect on the motion, when the point of support has a small oscillatory motion proportional to the horizontal oscillation of the ball of the pendulum.

7. Explain how a person sitting on a chair is able to move the chair across the room by a series of jerks, and without touching the ground with his feet.

8. A rectangle whose opposite sides AD, BC are vertical rests on a perfectly smooth horizontal table. A ring P rests on a smooth horizontal wire joining the middle points E and F of AD, BC at a distance C from BC. On a sudden the rod BC becomes repulsive and drives the ring towards AD. Find the velocity of the ring just before it strikes AD, and the space through which the rectangle has moved.

9. An elastic string is rolled without tension round a perfectly rough cylinder. One end of the string being attached to a fixed point, the cylinder descends by its own weight. Supposing the centre of the cylinder to describe a straight line, prove that

$$y + \frac{x}{2} = \tfrac{1}{2} g t^2,$$

where x and y are respectively the unstretched and stretched lengths of the string unrolled from the cylinder.

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